JEE Advanced (2022) PAPER-I

Mathematics

SECTION 1 (Maximum Marks: 24)

- This section contains **EIGHT (08)** questions.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numertic keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate**/**round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** if the correct numerical value is entered;

Zero Marks : 0 In all other cases.

Q. 1. Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

Q. 2. Let α be a positive real number. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: (\alpha, \infty) \to \mathbb{R}$ be the functions defined by

$$f(x) = \sin \frac{\pi x}{12} \text{ and } g(x) = \frac{2\log_e(\sqrt{x}-\sqrt{\alpha})}{\log_e(e^{\sqrt{x}}-e^{\sqrt{\alpha}})}.$$

Then the value of $\lim_{x\to\alpha^+} f(g(x))$ is ....

Q. 3. In a study about a pandemic, data of 900 persons was collected. It was found that

190, persons had symptom of fever,

220, persons had symptom of cough,

220, persons had symptom of breathing problem,

330, persons had symptom of fever or cough or both,

350, persons had symptom of cough or breathing problem or both,

340, persons had symptom of fever or breathing problem or both.

30, persons had all three symptoms (fever, cough and breathing problem).

If a person is chosen randomly from these 900 persons, then the probability that the person has at most one symptom is ....

Q. 4. Let z be a complex number with non-zero imaginary part. If

$$\frac{2+3z+4z^2}{2-3z+4z^2}$$

is a real number, then the value of $|z|^2$ is ....

Q. 5. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation

$\bar{z} - z^2 = i(\bar{z} + z^2)$ is ....

Q. 6. Let $l_1, l_2, ..., l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, ..., w_{100}$ be ocnsecutive terms of another arithmetic progression with common difference d_2, where $d_1d_2 = 10$. For each $i = 1, 2, ..., 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ....

Q. 7. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is ....

Q. 8. Let ABC be the triangle with $AB = 1$, $AC = 3$ and $\angle BAC = \frac{\pi}{2}$. If a circle of radius $r > 0$ touches the sides AB, AC and also touches internally the circumcircle of the triangle ABC, then the value of r is ....

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 **ONLY** if (all) the correct option(s) is (are) chosen;

Partial Marks : +3 If all the four options are correct but **ONLY** three options are chosen;

Partial Marks : +2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;

Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e., the question is unanswered);

Negative Marks : –2 In all other cases.

Q. 9. Consider the equation

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}dx = 1, a \in (-\infty, 0) \cup (1, \infty).$$

Which of the following statements is/are TRUE?

(A) No a satisfies the above equation

(B) An integer a satisfies the above equation

(C) An irrational number a satisfies the above equation

(D) More than one a satisfy the above equation

Q. 10. Let $a_1, a_2, a_3, ...$ be an arithmetic progression with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, ...$ be such that $T_1 = 3$ and $T_{n+1} - Tn = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE ?

(A) $T_{20} = 1604$

(B) $\Sigma_{k=1}^{20} T_k=10510$

(C) $T_{30} = 3454$

(D) $\Sigma_{k=1}^{30} T_k=35610$

Q. 11. Let P_1 and P_2 be two planes given by

$P_1 : 10x + 15y + 12z - 60 = 0,$

$P_2 : -2x + 5y + 4z - 20 = 0.$

Which of the following straight lines can be an edge of some tetrahedron whose two faces lie on P_1 and P_2 ?

(A) $\frac{x-1}{0}=\frac{y-1}{0}=\frac{z-1}{5}$

(B) $\frac{x-6}{-5}=\frac{y}{2}=\frac{z}{3}$

(C) $\frac{x}{-2}=\frac{y-4}{5}=\frac{z}{4}$

(D) $\frac{x}{1}=\frac{y-4}{-2}=\frac{z}{3}$

Q. 12. Let S be the reflection of a point Q with respect to the plane given by

$$\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$$

where t, p are real parameters and $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along the three positive coordinate axes. If the position vectors of Q and S are $10\hat{i} + 15\hat{j} + 20\hat{k}$ and $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ respectively, then which of the following is/are TRUE?

(A) $3(\alpha + \beta) = -101$

(B) $3(\beta + \gamma) = -71$

(C) $3(\gamma + \alpha) = -86$

(D) $3(\alpha + \beta + \gamma) = -121$

Q. 13. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ?

(A) $SQ_1 = 2$

(B) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$

(C) $PQ_1 = 3$

(D) $SQ_2 = 1$

Q. 14. Let $|M|$ denote the determinant of a square matrix M. Let $g: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be the function defined by

where $g(\theta) = \sqrt{f(\theta) - 1} + \sqrt{f\left(\frac{\pi}{2} - \theta\right) - 1}$

$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} \sin\pi & \cos\left(\theta + \frac{\pi}{4}\right) & \tan\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta + \frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}$$

Let $p(x)$ be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$, and $p(2) = 2 - \sqrt{2}$. Then, which of the following is/are TRUE ?

(A) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$

(B) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$

(C) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$

(D) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

SECTION 3 (Maximum Marks: 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists: **List-I** and **List-II**.
- **List-I** has **Four** entries (I), (II), (III) and (IV) and **List-II** has **Five** entries (P), (Q), (R), (S) and (T).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** if the option corresponding to the correct combination is chosen;

Zero Marks : 0 if none of the options is chosen (i.e., the question is unanswered);

Negative Marks : –1 in all other cases.

Q. 15. Consider the following lists:

List-I	**List-II**
(I) $\left\{x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]: \cos x + \sin x = 1\right\}$	**(P)** has two elements
(II) $\left\{x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right]: \sqrt{3}\tan 3x = 1\right\}$	**(Q)** has three elements
(III) $\left\{x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right]: 2\cos(2x) = \sqrt{3}\right\}$	**(R)** has four elements
(IV) $\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right]: \sin x - \cos x = 1\right\}$	**(S)** has five elements
	(T) has six elements

The correct option is:

(A) (I) → (P); (II) → (S); (III) → (P); (IV) → (S)

(B) (I) → (P); (II) → (P); (III) → (T); (IV) → (R)

(C) (I) → (Q); (II) → (P); (III) → (T); (IV) → (S)

(D) (I) → (Q); (II) → (S); (III) → (P); (IV) → (R)

Q. 16. Two players, P_1 and P_2, play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If $x > y$, then P_1 scores 5 points and P_2 scores 0 point. If $x = y$, then each player scores 2 points. If $x < y$, then P_1 scores 0 point and P_2 scores 5 points. Let X_i and Y_i be the total scores of P_1 and P_2, respectively, after playing the i^{th} round.

List-I	**List-II**
(I) Probability of $(X_2 \geq Y_2)$ is	**(P)** $\frac{3}{8}$
(II) Probability of $(X_2 > Y_2)$ is	**(Q)** $\frac{11}{16}$
(III) Probability of $(X_3 = Y_3)$ is	**(R)** $\frac{5}{16}$
(IV) Probability of $(X_3 > Y_3)$ is	**(S)** $\frac{355}{864}$
	(T) $\frac{77}{432}$

The correct option is:

(A) (I) → (Q); (II) → (R); (III) → (T); (IV) → (S)

(B) (I) → (Q); (II) → (R); (III) → (T); (IV) → (T)

(C) (I) → (P); (II) → (R); (III) → (Q); (IV) → (S)

(D) (I) → (P); (II) → (R); (III) → (Q); (IV) → (T)

Q. 17. Let p, q, r be nonzero real numbers that are, respectively, the 10^{th}, 100^{th} and 1000^{th} terms of a harmonic progression. Consider the system of linear equations

$$x + y + z = 1$$
$$10x + 100y + 1000z = 0$$
$$qr\, x + pr\, y + pq\, z = 0$$

List-I	**List-II**
(I) If $\frac{q}{r} = 10$, then the system of linear equations has	**(P)** $x = 0, y = \frac{10}{9}, z = -\frac{1}{9}$ as a solution
(II) If $\frac{p}{r} \neq 100$, then the system of linear equations has	**(Q)** $x = \frac{10}{9}, y = -\frac{1}{9}, z = 0$ as a solution
(III) If $\frac{p}{q} \neq 10$, then the system of linear equations has	**(R)** infinitely many solutions
(IV) If $\frac{p}{q} = 10$, then the system of linear equations has	**(S)** no solution
	(T) at least one solution

The correct option is:

(A) (I) → (T); (II) → (R); (III) → (S); (IV) → (T)

(B) (I) → (Q); (II) → (S); (III) → (S); (IV) → (R)

(C) (I) → (Q); (II) → (R); (III) → (P); (IV) → (R)

(D) (I) → (T); (II) → (S); (III) → (P); (IV) → (T)

Q. 18. Consider the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Let $H(\alpha, 0)$, $0 < \alpha < 2$, be a point. A straight line drawn through H parallel to the y-axis crosses the ellipse and its auxiliary circle at points E and F respectively, in the first quadrant. The tangent to the ellipse at the point E intersects the positive x-axis at a point G. Suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis.

List-I	List-II
(I) If $\phi = \frac{\pi}{4}$, then the area of the triangle FGH is	**(P)** $\frac{\left(\sqrt{3}-1\right)^4}{8}$
(II) If $\phi = \frac{\pi}{3}$, then the area of the triangle FGH is	**(Q)** 1
(III) If $\phi = \frac{\pi}{6}$, then the area of the triangle FGH is	**(R)** $\frac{3}{4}$
(IV) If $\phi = \frac{\pi}{12}$, then the area of the triangle FGH is	**(S)** $\frac{1}{2\sqrt{3}}$
	(T) $\frac{3\sqrt{3}}{2}$

The correct option is:

(A) (I) → (R); (II) → (S); (III) → (Q); (IV) → (P)

(B) (I) → (R); (II) → (T); (III) → (S); (IV) → (P)

(C) (I) → (Q); (II) → (T); (III) → (S); (IV) → (P)

(D) (I) → (Q); (II) → (S); (III) → (Q); (IV) → (P)

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Q. no.	Answer	Topic name	Chapter name
1	[2.36]	Properties of inverse Trigonometric functions	Inverse Trigonometric Funtions
2	[0.50]	Limits and Function	Limit and Derivatives
3	[0.80]	Algebra of Sets	Sets Theory
4	[0.50]	Properties of Modulas and Argument	Complex Number
5	[4]	Properties of Modulas and Argument	Complex Number
6	[18900]	Sum of the n^{th} term of A.P.	Sequence and Series
7	[569]	Permutations	Permutations and Combinations
8	[0.84]	Circle	Two Dimensional (2D)
9	C&D	Properties of definite integral	Integration
10	B&C	Sum of the n^{th} term of A.P.	Sequence and Series
11	A,B&D	Intersection of lines and planes	Lines and Planes
12	A,B&C	Points and Lines	Three Dimensional (3D)
13	B,C&D	Tangents and normals of Parabola	Parabola
14	A&C	Matrics and Determinants	Matrics and Determinants
15	B	Trigonometric Equations	Trigonometric Funtions
16	A	Probability	Probability
17	B	Solution of Linear Equations	Matrics and Determinants
18	C	Ellipse	Two Dimensional (2D)

Answers

1. Correct answer is [2.36]

Explanation:

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\,\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

Converting into $\tan^{-1}$ form

Let $\cos^{-1}\sqrt{\frac{2}{2+\pi^2}} = x$

$$\Rightarrow \sqrt{\frac{2}{2+\pi^2}} = \cos x$$

$$\Rightarrow \frac{2}{2+\pi^2} = \cos^2 x$$

$$\Rightarrow \frac{2+\pi^2}{2} = \sec^2 x$$

$$\Rightarrow \frac{\pi^2}{2} = \tan^2 x$$

$$\Rightarrow \tan^{-1}\frac{\pi}{\sqrt{2}} = x$$

and $\sin^{-1} x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$$\Rightarrow \sin^{-1}\frac{2\sqrt{2}\,\pi}{2+\pi^2} = \tan^{-1}\frac{\frac{2\sqrt{2}\,\pi}{2+\pi^2}}{\sqrt{1-\left(\frac{2\sqrt{2}\,\pi}{2+\pi^2}\right)^2}}$$

$$\Rightarrow = \tan^{-1}\frac{\frac{2\sqrt{2}\,\pi}{2+\pi^2}}{\frac{\sqrt{4+\pi^4+4\pi^2-8\pi^2}}{2+\pi^2}}$$

$$\Rightarrow = \tan^{-1}\frac{2\sqrt{2}\,\pi}{\pi^2-2}$$

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

$$\Rightarrow \frac{3}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{1}{4}\tan^{-1}\left(\frac{2\sqrt{2}\pi}{\pi^2-2}\right)+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

$$\Rightarrow \frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}-\frac{1}{4}\tan^{-1}\frac{2\sqrt{2}\pi}{2-\pi^2}$$

$$\left\{\because \tan^{-1}\frac{\sqrt{2}}{\pi}=\cos^{-1}\frac{\pi}{\sqrt{2}}\right\}$$

$$\Rightarrow \frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}-\frac{1}{4}\tan^{-1}\frac{\left(2.\frac{\pi}{\sqrt{2}}\right)}{\left\{1-\left(\frac{\pi}{\sqrt{2}}\right)^2\right\}}$$

$$\left\{\because \tan^{-1}\frac{\pi}{2}+\cot^{-1}\frac{\pi}{\sqrt{2}}=\frac{\pi}{2}\right\}$$

$$\Rightarrow \frac{\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)-\frac{1}{4}\left(-\pi+2\tan^{-1}\frac{\pi}{\sqrt{2}}\right)$$

$$\Rightarrow \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4} \cong 2.36$$

2. Correct answer is [0.50]

Explanation:

$$\lim_{x\to\alpha^+} g(x) = \lim_{x\to\alpha^+}\frac{\frac{2}{\sqrt{x}-\sqrt{\alpha}}\left(\frac{1}{2\sqrt{x}}\right)}{\frac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\left(\frac{1}{2\sqrt{x}}.e^{\sqrt{x}}\right)}$$

$$= \lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}{\sqrt{x}-\sqrt{\alpha}}.\frac{2}{e^{\sqrt{x}}}$$

$$= \lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}}.\frac{2}{e^{\sqrt{x}}}$$

$$= 2$$

$$\lim_{x\to\alpha^+} f(g(x)) = f\left(\lim_{x\to\alpha^+} g(x)\right)$$

$$= \sin\frac{\pi\times 2}{12}=\frac{1}{2}$$

$$= 0.50$$

3. Correct answer is [0.80]

Explanation: Let person had symptom of fever $= n(\text{F}) = 190$

Person had symptom of cough $= n(\text{C}) = 220$

Person had symptom of breathing $= n(\text{B}) = 220$

Person had symptom of fever or cough $= n(\text{F} \cup \text{C}) = 330$

Person had symptom of cough or breathing $= n(\text{C} \cup \text{B}) = 350$

Person had symptom of fever or breathing $= n(\text{F} \cup \text{B}) = 340$

Person had symptom of fever, cough and breathing $= n(\text{F} \cap \text{C} \cup \text{B}) = 30$

So $n(\text{F} \cap \text{C}) = n(\text{F}) + n(\text{C}) - n(\text{F} \cup \text{C})$

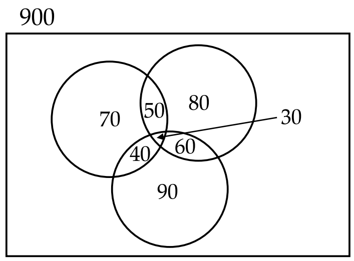
$= 190 + 220 - 330$

$= 80$

and $n(F \cap B) = n(F) + n(B) - n(F \cup B)$
$= 190 + 220 - 340$
$= 70$

and $n(C \cap B) = n(C) + n(B) - n(C \cap B)$
$= 220 + 220 - 350$
$= 90$

Number of people having at most one symptom
$= 70 + 80 + 90 + 480$
$= 720$



$$\text{Required probability} = \frac{720}{900} = 0.80$$

4. Correct answer is [0.50]

Explanation: Let
$$w = \frac{4z^2+3z+2}{4z^2-3z+2}$$
$$= 1 + \frac{6z}{4z^2-3z+2}$$
$$= 1 + \frac{6}{2\left(2z+\frac{1}{z}\right)-3}$$

$\because\ w \in R$ then $2z + \frac{1}{z} \in R$

$$\Rightarrow \quad 2z + \frac{1}{z} = 2\bar{z}+\frac{1}{\bar{z}}$$
$$\Rightarrow \quad 2(z-\bar{z}) = \frac{1}{\bar{z}}-\frac{1}{z}$$
$$\Rightarrow \quad 2(z-\bar{z}) = \frac{z-\bar{z}}{|z|^2}$$
$$\Rightarrow \quad 2(z-\bar{z}) - \frac{z-\bar{z}}{|z|^2} = 0$$
$$\Rightarrow \quad 2(z-\bar{z}) = \left(2-\frac{1}{|z|^2}\right) = 0$$

$\because$ $z = \bar{z}$

but $z \neq \bar{z}$ given

So $2-\dfrac{1}{|z|^2} = 0$

$$|z|^2 = \frac{1}{2} = 0.5$$

Alternative method:
$$w = 4z^2 + 3z + 2$$
$$= 1+\frac{6}{4z+\frac{2}{z}-3}$$
$$\Rightarrow \quad 4z + \frac{2}{z} = \text{Real}$$
$$\Rightarrow \quad 4(x+iy) + \frac{2}{x+iy} = \text{Real} \quad [\because z = x+iy]$$
$$\Rightarrow \quad 4(x+iy) + \frac{2(x-iy)}{x^2+y^2} = \text{Real}$$

Im = 0
$$4y - \frac{2y}{x^2+y^2} = 0$$
$$\Rightarrow \quad 4y = \frac{2y}{x^2+y^2}$$
$$\Rightarrow \quad x^2+y^2 = \frac{1}{2}$$
$$|z|^2 = \frac{1}{2} = 0.5$$

5. Correct answer is [4]

Explanation: Given:
$$\bar{z} - z^2 = i(\bar{z} + z^2)$$
$$(1-i)\,\bar{z} = (1+i)\,z^2$$
$$\Rightarrow \quad \frac{1-i}{1+i}\bar{z} = z^2$$
$$\Rightarrow \quad \left(\frac{-2i}{2}\right)\bar{z} = z^2$$
$$\Rightarrow \quad z^2 = -i\bar{z}$$

Let $z = x + iy$

$\therefore\ (x^2-y^2) + i(2xy) = -i(x-iy)$

So $x^2 - y^2 + y = 0$ …(1)

$(2y+1)x = 0$ …(2)

$$\Rightarrow \quad x = 0 \text{ or } y = -\frac{1}{2}$$

Case I: When $x = 0$

$\therefore$ (1) $\Rightarrow y(1-y) = 0 \Rightarrow y = 0, 1$

$\therefore$ (0, 0), (0, 1)

Case II: When $y = -\frac{1}{2}$

$$\therefore (1) \Rightarrow \quad x^2 - \frac{1}{4} - \frac{1}{2} = 0$$
$$\Rightarrow \quad x^2 = \frac{3}{4}$$
$$\Rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$$

$\therefore \quad \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right), \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$

Number of distinct 'z' is equal to 4.

6. Correct answer is [18900]

Explanation: For A.P. $l_1, l_2 \ldots\ldots l_{100}$

Let $T_1 = a$ and common diff $= d_1$ and similarly

For A.P. $w_1, w_2 = w_{100}$

$T_1 = b$ and common diff. $= d_2$

given $\quad A_{51} - A_{50} = l_{51}w_{51} - l_{50}w_{50} = 1000$

$= (a + 50d_1)(b + 50d_2) - (a + 49d_1)(b + 49d_2) = 100$

$= 50bd_1 + 50ad_2 + 2500d_1d_2 - 49ad_2 - 49bd_1 - 2401d_1d_2 = 1000$

$bd_1 + ad_2 + 99d_1d_2 = 100$

$\therefore \quad bd_1 + ad_2 = 10 \ldots(1) \quad \text{As } (d_1d_2 = 10)$

$\therefore \quad A_{100} - A_{90} = l_{100}w_{100} - l_{90}w_{90}$

$= (a + 99d_1)(b + 99d_2) - (a + 89d_1)(b + 89d_2)$

$= 99bd_1 + 99ad_2 + 99^2d_1d_2 - 89bd_1 - 89ad_2 - 89^2d_1d_2$

$= 10(bd_1 + ad_2) + 1880d_1d_2$

$= 10(10) + 18800$

$= 18900$

7. Correct answer is [569]

Explanation: Counting integers starting from 2

Case - I : If zero on 2^{nd} place

i.e., $\quad 202\underset{5}{\uparrow} = 5$ cases

or $\quad 20\underset{4}{\uparrow}\underset{6}{\uparrow} = 24$ cases

Case-II: If non-zero number on 2^{nd} place

i.e., $\quad 2\underset{5}{\uparrow}\underset{6}{\uparrow}\underset{6}{\uparrow} = 180$ cases

Counting integers starting from 3

$3\underset{6}{\uparrow}\underset{6}{\uparrow}\underset{6}{\uparrow} = 216$ cases

Counting integers starting from 4

Case- I: If 0, 2, or 3 on 2^{nd} place

i.e., $\quad 4\underset{3}{\uparrow}\underset{6}{\uparrow}\underset{6}{\uparrow} = 108$ cases

Case-II: If 4 on 2^{nd} place

i.e., $\quad 44\underset{6}{\uparrow}\underset{6}{\uparrow} = 36$ cases

Therefore total numbers

$= 5 + 24 + 180 + 216 + 108 + 36$

$= 569$

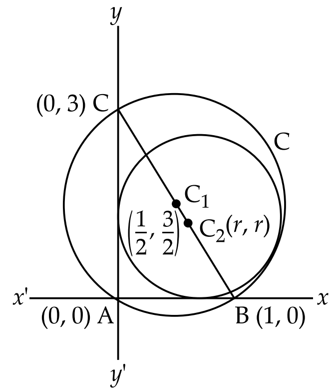
8. Correct answer is [0.84]

Explanation: Let A be the origin and B lies on X-axis,

C on Y axis as $C_1 = \left(\frac{1}{2}, \frac{3}{2}\right)$ and $C_2 = (r, r)$

$\therefore$ Equation of circumcircle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{5}{2} \quad \ldots(1)$$



Required circle touches AB and AC, have radius r

$\Rightarrow$ Equation be $(x - r)^2 + (y - r)^2 = r^2 \quad \ldots(2)$

If circle in equation (2) touches circumcircle internally, we have

$$d_{c_1c_2} = |r_1 - r_2|$$

$$\Rightarrow \quad \left(\frac{1}{2} - r\right)^2 + \left(\frac{3}{2} - r\right)^2 = \left(\left|\sqrt{\frac{5}{2}} - r\right|\right)^2$$

$$\Rightarrow \frac{1}{4} + r^2 - r + \frac{9}{4} + r^2 - 3r = \frac{5}{2} + r^2 - \sqrt{10}\, r$$

$$\Rightarrow 2r^2 - 4r + \frac{5}{2} = \frac{5}{2} + r^2 - \sqrt{10}\, r$$

$$\Rightarrow \quad r^2 - 4r + \sqrt{10}\, r = 0$$

$r = 0$ (reject)

$r = 4 - \sqrt{10}$

$r = 0.837$

9. Options (C) & (D) are correct

Explanation: Let $\quad I = \int_1^e \frac{(\log_e^x)^{1/2}\, dx}{x\,(a - (\log_e^x)^{3/2})^2} = 1$

Let $\quad (a - \log_e x)^{3/2} = t$

$$\Rightarrow \quad -\frac{3}{2}(\log_e x)^{3/2} \cdot \frac{1}{x}\, dx = dt$$

$$I = \int_a^{a-1} \frac{\left(-\frac{2}{3}\right) dt}{t^2} = 1$$

$$\begin{Bmatrix} x = 1 & t = a \\ x = e & t = a - 1 \end{Bmatrix}$$

$$= \frac{-2}{3}\left[\frac{t^{-1}}{-1}\right]_a^{a-1} = 1$$

$$= \frac{2}{3}\left(\frac{1}{(a-1)} - \frac{1}{a}\right) = 1$$

$$= \frac{2}{3}\frac{1}{a(a-1)} = 1$$

$$= 3a^2 - 3a - 2 = 0$$

$$= a = \frac{3 \pm \sqrt{33}}{6}$$

10. Options (B) & (C) are correct

Explanation: Here $a_n = 7 + (n-1)8$ and $T_1 = 3$, $a_1 = 7, d = 8$

Also, $T_{n+1} = T_n + a_n$

$T_n = T_{n-1} + a_{n-1}$

$T_2 = T_1 + a_1$

$\therefore$ $T_{n+1} = (T_{n-1} + a_{n-1}) + a_n$

$T_{n+1} = T_{n-2} + a_{n-2} + a_{n-1} + a_n$

So $T_{n+1} = T_1 + a_1 + a_2 \ldots\ldots a_n$

$$T_{n+1} = T_1 + \frac{n}{2}(2 \times 7 + (n-1)8)$$

$$T_{n+1} = T_1 + n(4n+3) \quad \ldots(1)$$

For (A), if $n = 19$ $T_{20} = 3 + (19)(79) = 1504$

For (C), if $n = 29$ $T_{30} = 3 + 29(119) = 3454$

For (B), $\sum_{k=1}^{20} T_k = \sum_{k=1}^{20} (T_1 + 4n^2 + 3n) + 3$

$$= 3 + \sum_{k=1}^{20} (3 + 4n^2 + 3n) \quad (\therefore T_1 = 3)$$

$$= 3 + 3(19) + \frac{3(19)(20)}{2} + \frac{4(19)(20)(34)}{6}$$

$$= 3 + 10507 = 10510$$

Similarly, for (D)

$$\sum_{k=1}^{30} T_k = 3 + \sum_{k=1}^{29} (4n^2 + 3n + 3) = 35615$$

11. Options (A), (B) & (D) are correct

Explanation: P_1 and P_2 be two planes given by

$P_1 : 10x + 15y + 12z - 60 = 0$

$P_2 : -2x + 5y + 4z - 20 = 0$

Now finding line of intersection of both the planes,

Let $z = \lambda$

then $10x + 15y = 60 - 12\lambda \quad \ldots(1)$

$-2x + 5y = 20 - 4\lambda \quad \ldots(2)$

Now solving the eq. (1) and (2) we get,

$$\frac{x}{0} = \frac{y-4}{-4} = \frac{z}{5} \quad (\because \lambda = z)$$

Now any skew line with the line of intersection of given plane can be edge of tertrahedron.

Now using above concept we will solve all options.

For option (A)

$$\frac{x-1}{0} = \frac{y-1}{0} = \frac{z-1}{5}$$

Point is $(1, 1, 5\lambda + 1)$

Now satisfying this point in given plane we have,

$10 \times 1 + 15 \times 1 + 12 \times (5\lambda + 1) - 60 = 0$

$\Rightarrow \quad 60\lambda = 23$

$\Rightarrow \quad \lambda = \frac{23}{60}$

Now we can see line is intersecting the plane P_1, at some point.

Now checking for plane (P_2)

$-2 \times 1 + 5 \times 1 + 4(5\lambda + 1) = 20$

$\Rightarrow \quad 20\lambda = 13$

$\lambda = \frac{13}{20}$

Also intersecting plane (P_2)

Hence, it can be the edge of tetrahedron.

For option (B)

$\frac{x-6}{-5} = \frac{y}{2} = \frac{z}{3}$ point is $(-5\lambda + 6, 2\lambda, 3\lambda)$

this point is sytisfying plane P_1

$10(-5\lambda + 6) + 15 \times 2\lambda + 12 \times 3\lambda = 60$

$\Rightarrow \quad -50\lambda + 60 + 30\lambda + 36\lambda = 60$

$\Rightarrow \quad 16\lambda = 0$

$\lambda = 0$

Now checking for plane P_2

$-2(-5\lambda + 6) + 5 \times 2\lambda + 4 \times 3\lambda = 20$

$\Rightarrow \quad + 10\lambda - 12 + 10\lambda + 12\lambda = 20$

$\Rightarrow \quad 32\lambda = 32$

$\Rightarrow \quad \lambda = 1$

Hence, it can be the edge of tetrahedron.

For option (C)

$$\frac{x}{-2} = \frac{y-4}{5} = \frac{z}{4}$$

point is $(-2\lambda, 5\lambda + 4, 4\lambda)$

Similarly checking in plane P_1 we get.

$$\lambda \text{ for } P_1 = 0$$
$$\lambda \text{ for } P_2 = 0$$

Hence, it can not be the edge of tetrahedron

For option (D),

$$\frac{x}{1} = \frac{y-4}{-2} = \frac{z}{3}$$

point $(\lambda, -2\lambda + 4, 3\lambda)$ and for $\lambda = 0$ point will be $(0, -4, 0)$ which is lying on line of intersection and DR of plane P_2 is $(-2, 5, 4)$ and DR of line is $(1, -2, 3)$

Now line is lying completely on P_2

Hence, it can be the edge of tetrahedron.

12. Options (A), (B) & (C) are correct

Explanation: Given: Eq. of plane

$$\vec{r} = -(t+p)\,\hat{i} + t\,\hat{j} + (1+p)\,\hat{k}$$

on rearranging we get,

$$\vec{r} = k + t + (-\hat{i} + \hat{j}) + p(-\hat{i} + \hat{k})$$

So, eq. of plane in standard form is given by

$$[\vec{r} - \hat{k} \quad -\hat{i} + \hat{j} \quad -\hat{i} + \hat{k}] = 0$$

$$\Rightarrow \begin{bmatrix} x & y & z-1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad x + y + z = 1 \qquad \ldots(1)$$

Now given co-ordinate of Q = (10, 15, 20)

And Co-ordinates of S = (α, β, γ)

Now using the image formula of point and plane we get

$$\frac{\alpha-10}{1} = \frac{\beta-15}{1} = \frac{\gamma-20}{1} = \frac{-2(10+15+20-1)}{3}$$

$$\Rightarrow \quad \alpha - 10 = \beta - 15 = \gamma - 20 = \frac{-88}{3}$$

$$\Rightarrow \quad \alpha = \frac{-58}{3}, \beta = \frac{-43}{3}, \gamma = \frac{-28}{3}$$

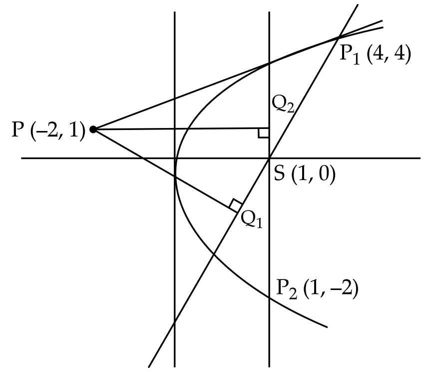
Now solving all options.

$$3(\alpha + \beta) = -101$$
$$3(\beta + \gamma) = -71$$
$$3(\gamma + \alpha) = -86$$
$$3(\alpha + \beta + \gamma) = -129$$

13. Options (B), (C) & (D) are correct

Explanation: Let $P_1 (t^2, 2t)$ then tangent at P_1 will be

$$ty = x + t^2$$



Since, it passes through (–2, 1)

$$\Rightarrow \quad t^2 - t - 2 = 0$$
$$\Rightarrow \quad t = 2, -1$$

So, we get point P_1 (4, 4) and P_2 (1, –2)

Now finding the equation of $SP_1 : 4x - 3y - 4 = 0$

And equation of $SP_2 : x - 1 = 0$

Now finding the point by foot of point on line formula,

$$Q_1: \frac{x_1+2}{4} = \frac{y_1-1}{-3} = \frac{-(-8-3-4)}{25} = \frac{3}{5}$$

We get $x_1 = \frac{2}{5}, y_1 = \frac{-4}{5}$ and $Q_2 = (1, 1)$

Now using the distance formula we get,

$$SQ_1 = \sqrt{\left(1-\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

$$Q_1Q_2 = \sqrt{\frac{9}{25} + \frac{81}{25}} = \frac{3\sqrt{10}}{5}$$

$$PQ_1 = \sqrt{\frac{144}{25} + \frac{81}{25}} = 3$$

$$SQ_2 = 1$$

Hence, option (B, C, D) are correct.

14. Options (A) & (C) are correct.

Explanation: Given:

$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$$+ \frac{1}{2}\begin{vmatrix} \sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}$$

$$\Rightarrow \quad f(\theta) = \frac{1}{2}\begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix}$$

$$+ \frac{1}{2}\begin{vmatrix} 0 & -\sin\left(\theta-\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & 0 & \log_e\frac{4}{\pi} \\ -\tan\left(\theta-\frac{\pi}{4}\right) & -\log\left(\frac{4}{\pi}\right) & 0 \end{vmatrix}$$

$$\Rightarrow \quad f(\theta) = \frac{1}{2}\begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix} + 0$$

(Skew symetric)

$\Rightarrow \quad f(\theta) = 1 + \sin^2\theta$

So $\quad g(\theta) = |\sin\theta| + |\cos\theta|$

$g(\theta) = [1, \sqrt{2}\,]$

for $\theta \in \left[0, \frac{\pi}{2}\right]$

Again let $\quad p(x) = k(x-\sqrt{2})(x-1)$

$2-\sqrt{2} = k(2-\sqrt{2})(2-1)$

$\Rightarrow \quad k = 1 \quad \therefore [p(z) = 2-\sqrt{2}\,]$

$\therefore \quad p(x) = (x-\sqrt{2})(x-1)$

For option (A) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$ Correct.

For option (B) $p\left(\frac{1+3\sqrt{2}}{4}\right) < 0$ Incorrect.

For option (C) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$ Correct.

For option (D) $p\left(\frac{5-\sqrt{2}}{4}\right) > 0$ Incorrect.

15. Options (B) is correct.

Explanation: Solving all question one by one we get,

(i) $\left\{x \in \left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right], \cos x + \sin x = 1\right\}$

$\cos x + \sin x = 1$

$\Rightarrow \quad \sin\left(\frac{\pi}{4}+x\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{\pi}{4} + x = n\pi + (-1)^n\frac{\pi}{4}$

$\Rightarrow \quad x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$

So, $x \in \left\{0, \frac{\pi}{2}\right\}$

$\therefore$ x has 2 elements $\rightarrow$ 0

(ii) $\left\{x \in \left(\frac{-5\pi}{18}, \frac{5\pi}{18}\right), \sqrt{3}\tan 3x = 1\right\}$

$\Rightarrow \quad \tan 3x = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad x = n\pi + \frac{\pi}{6}$

$\Rightarrow \quad x = \frac{n\pi}{3} + \frac{\pi}{18}$

So $\quad x \in \left\{\frac{\pi}{18}, \frac{-5\pi}{18}\right\}$

$\therefore$ x has 2 elements $\rightarrow p$

(iii) $\left\{x \in \left[\frac{-6\pi}{5}, \frac{6\pi}{5}\right], 2\cos 2x = \sqrt{3}\right\}$

$\Rightarrow \quad 2\cos 2x = \sqrt{3}$

$\Rightarrow \quad \cos 2x = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad 2x = 2n\pi \pm \frac{\pi}{6}$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{12}$

So $x \in \left\{\pm\frac{\pi}{12}, \pi \pm \frac{\pi}{12}, -\pi \pm \frac{\pi}{12}\right\}$

$\therefore$ x has 6 elements $\rightarrow$ T

(iv) $\left\{x \in \left[\frac{-7\pi}{4}, \frac{7\pi}{4}\right], \sin x - \cos x = 1\right\}$

$\Rightarrow \quad \sin x - \cos x = 1$

$\Rightarrow \quad \sin\left(x-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x - \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}$

$\Rightarrow \quad x = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{4}$

So, $\quad x \in \left\{\frac{\pi}{2}, \frac{-3\pi}{2}, \pi, -\pi\right\}$

$\therefore$ x has 4 elements $\rightarrow$ R

16. Options (A) is correct

Explanation: Given:

$$P\text{ (draw in 1 round)} = \frac{6}{36} = \frac{1}{6}$$

$$P\text{ (win in 1 round)} = \frac{1}{2}\left(1-\frac{1}{6}\right) = \frac{5}{12}$$

$$P\text{ (loss in 1 round)} = \frac{5}{12}$$

Now finding the probability of all we get.

$$P(X_2 > Y_2) = P(10, 0) + P(7, 2)$$
$$= \frac{5}{12}\times\frac{5}{12}+\frac{5}{12}\times\frac{1}{6}\times 2$$
$$= \frac{45}{144} = \frac{5}{16} \rightarrow R$$

$$P(X_2 = Y_2) = P(5, 5) + P(4, 4)$$
$$= \frac{5}{12}\times\frac{5}{12}\times 2+\frac{1}{6}\times\frac{1}{6}$$
$$= \frac{25+2}{72} = \frac{3}{8}$$

$$P(X_3 = Y_3) = P(6, 6) + P(7, 7)$$
$$= \frac{1}{6\times 6\times 6} + \frac{5}{12}\times\frac{1}{6}\times\frac{5}{12}\times\frac{1}{6}$$
$$= \frac{77}{432} \rightarrow F$$

$$P(X_3 > Y_3) = \frac{1}{2}\left(1-\frac{77}{432}\right) = \frac{355}{864} \rightarrow S$$

$$P(X_2 \geq Y_2) = \frac{5}{16}+\frac{3}{8} = \frac{11}{16} \rightarrow Q$$

17. Options (B) is correct

Explanation: Given:

$$x + y + z = 1 \quad \ldots(1)$$
$$10x + 100y + 1000z = 0 \quad \ldots(2)$$
$$qrx + pry + pqz = 0 \quad \ldots(3)$$

Now equation (3) can be re-written as

$$\frac{x}{p} = \frac{y}{q} = \frac{z}{r} = 0 \qquad \because \{p, q, r \neq 0\}$$

Now given p, q and r are 10^{th}, 100^{th} and 1000^{th} term of an. *h.p.*,

So let $p = \frac{y}{a+9d}$, $q = \frac{1}{a+99d}$, $r = \frac{1}{a+999d}$

Now from equation (3)

$$(a + 9d)x + (a + 99d)y + (a + 999d)z = 0$$

Now from eq. (1), (2) & (3) we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 1000 \\ a+9d & a+99d & a+999d \end{vmatrix} = 0$$

$$\Delta x = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 100 & 1000 \\ 0 & a+99d & a+999d \end{vmatrix} = 900(d-a)$$

$$\Delta y = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 0 & 1000 \\ a+9d & 0 & a+999d \end{vmatrix} = 990(a-d)$$

$$\Delta z = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 0 \\ a+9d & a+99d & 0 \end{vmatrix} = 90(d-a)$$

(I) If $\frac{q}{r} = 10 \Rightarrow a = d$

$$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$$

And eq. (1) and eq. (2) represents non-parallel plane eq. (2) and eq. (3) represents same plane $\Rightarrow$ Infinitely many solutions.

Now finding solution by taking $z = \lambda$

then $x + y = 1 - \lambda$

$$x + 10y = -100\lambda$$
$$x = \frac{10}{9} + 100\lambda$$
$$y = \frac{-1}{9} - 11\lambda$$

$\Rightarrow$ $(x, y, z) \in \left(\frac{10}{9}+10\lambda,\ \frac{-1}{9}-11\lambda,\ \lambda\right)$

So P is not valid for any value of $\lambda \rightarrow Q$

(II) $\frac{p}{r} \neq 100 \Rightarrow a \neq d$

$$\Delta = 0 \;\&\; \Delta_x, \Delta_y, \Delta_z \neq 0$$

So no solution.

II $\rightarrow$ S

(III) If $\frac{p}{q} \neq 10 \Rightarrow a \neq d$ then $\Delta_z \neq 0$

So no solution.

III $\rightarrow$ S

(IV) If $\frac{p}{q} = 10$

$\Rightarrow a = d$ then $\Delta_z = 0 \Rightarrow \Delta_x = \Delta_y = 0$

So infinitely many solutions.

IV $\rightarrow$ R

18. Options (C) is correct

Explanation: Given:

$$\frac{x^2}{4}+\frac{y^2}{3} = 1$$

Let $\alpha = 2\cos\phi$

Tangent at E$(2\cos\phi, \sqrt{3}\sin\phi)$

to the ellipse is $\frac{x\cos\phi}{2}+\frac{y\sin\phi}{\sqrt{3}} = 1$

This intersect x-axis at G$(2\sec\phi, 0)$

Area of triangle FGH $= \frac{1}{2}(2\sec\phi - 2\cos\phi)\, 2\sin\phi$

$$\Delta = 2\sin^2\phi.\tan\phi$$

$$\Delta = (1-\cos 2\phi).\tan\phi$$

I. If $\phi = \frac{\pi}{4}, \Delta = 1 \rightarrow Q$

II. If $\phi = \frac{\pi}{3}, \Delta = 2\left(\frac{\sqrt{3}}{2}\right)^2.\sqrt{3}$

$$= \frac{3\sqrt{3}}{2} \rightarrow T$$

III. If $\phi = \frac{\pi}{6}, \Delta = 2\left(\frac{1}{2}\right)^2.\frac{1}{\sqrt{3}}$

$$= \frac{1}{2\sqrt{3}} \rightarrow S$$

IV. If $\phi = \frac{\pi}{12}, \Delta = \left(1-\frac{\sqrt{3}}{2}\right).(2-\sqrt{3})$

$$= \frac{(\sqrt{3}-1)^4}{8} \rightarrow P$$

❑❑