JEE (Main) MATHEMATICS SOLVED PAPER

2021

26th August Shift 1

Time : 1 Hour **Total Marks : 100**

General Instructions :

1. *In Chemistry Section, there are 30 Questions (Q. no. 1 to 30).*
2. *In Chemistry, Section A consists of 20 multiple choice questions & Section B consists of 10 numerical value type questions. In Section B, candidates have to attempt any five questions out of 10.*
3. *There will be only one correct choice in the given four choices in Section A. For each question for Section A, 4 marks will be awarded for correct choice, 1 mark will be deducted for incorrect choice questions and zero mark will be awarded for not attempted question.*
4. *For Section B questions, 4 marks will be awarded for correct answer and zero for unattempted and incorrect answer.*
5. *Any textual, printed or written material, mobile phones, calculator etc. is not allowed for the students appearing for the test.*
6. *All calculations / written work should be done in the rough sheet is provided with Question Paper.*

Mathematics

Section A

Q. 1. If the sum of an infinite GP a, ar, ar^2, ar^3, ……. is 15 and the sum of the squares of its each term is 150, then the sum of ar^2, ar^4, ar^6, …. is :

(1) $\frac{1}{2}$ **(2)** $\frac{2}{5}$

(3) $\frac{25}{2}$ **(4)** $\frac{9}{2}$

Q. 2. Let ABC be a triangle with A (–3, 1) and $\angle ACB = \theta, 0 < \theta < \frac{\pi}{2}$. If the equation of the median through B is $2x + y - 3 = 0$ and the equation of angle bisector of C is $7x - 4y - 1 = 0$, then $\tan\theta$ is equal to :

(1) $\frac{4}{3}$ **(2)** $\frac{1}{2}$

(3) 2 **(4)** $\frac{3}{4}$

Q. 3. Let $\theta \in \left(0, \frac{\pi}{2}\right)$. If the system of linear equations,

$(1 + \cos^2\theta)\, x + \sin^2\theta\, y + 4 \sin 3\theta\, z = 0$

$\cos^2\theta\, x + (1 + \sin^2\theta)\, y + 4 \sin 3\theta\, z = 0$

$\cos^2\theta\, x + \sin^2\theta\, y + (1 + 4 \sin 3\theta)\, z = 0$

has a non-trivial solution, then the value of θ is:

(1) $\frac{4\pi}{9}$ **(2)** $\frac{\pi}{18}$

(3) $\frac{5\pi}{18}$ **(4)** $\frac{7\pi}{18}$

Q. 4. The sum of solutions of the equation $\frac{\cos x}{1+\sin x} = |\tan 2x|$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}, -\frac{\pi}{4}\right\}$ is:

(1) $-\frac{11\pi}{30}$ **(2)** $-\frac{7\pi}{30}$

(3) $-\frac{\pi}{15}$ **(4)** $\frac{\pi}{10}$

Q. 5. Let A and B be independent events such that $P(A) = p$, $P(B) = 2p$. The largest value of p, for which P (exactly one of A, B occurs) = $\frac{5}{9}$, is:

(1) $\frac{1}{4}$ **(2)** $\frac{2}{9}$

(3) $\frac{1}{3}$ **(4)** $\frac{5}{12}$

Q. 6. If the truth value of the Boolean expression $((p \vee q) \wedge (q \rightarrow r) \wedge (\sim r)) \rightarrow (p \wedge q)$ is false, then the truth values of the statements p, q, r respectively can be:

(1) FFT **(2)** FTF

(3) TFT **(4)** TFF

Q. 7. The value of $\lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2+4r^2}$ is:

(1) $\frac{1}{4}\tan^{-1}(4)$ **(2)** $\tan^{-1}(4)$

(3) $\frac{1}{4}\tan^{-1}(2)$ **(4)** $\frac{1}{2}\tan^{-1}(4)$

Q. 8. Let $y = y(x)$ be a solution curve of the differential equation $(y + 1)\tan^2 x\, dx + \tan x\, dy + y dx = 0$, $x \in \left(0, \frac{\pi}{2}\right)$. If $\lim_{x\to 0^+} x\, y(x) = 1$, then the value of $y\left(\frac{\pi}{4}\right)$ is:

(1) $\frac{\pi}{4}-1$ **(2)** $\frac{\pi}{4}+1$

(3) $\frac{\pi}{4}$ **(4)** $-\frac{\pi}{4}$

Q. 9. The mean and standard deviation of 20 observations were calculated as 10 and 2.5 respectively.

It was found that by mistake one data value was taken as 25 instead of 35. If α and $\sqrt{\beta}$ are the mean and standard deviation respectively for correct data, then (α, β) is:

(1) (11, 25) **(2)** (11, 26)

(3) (10.5, 26) **(4)** (10.5, 25)

Q. 10. Let $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$, $0 < x < 1$. Then:

(1) $(1 + x)^2 f'(x) + 2(f(x))^2 = 0$

(2) $(1 - x)^2 f'(x) + 2(f(x))^2 = 0$

(3) $(1 + x)^2 f'(x) - 2(f(x))^2 = 0$

(4) $(1 - x)^2 f'(x) - 2(f(x))^2 = 0$

Q. 11. The sum of the series

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{2^2}{x^4+1}+.....+\frac{2^{100}}{x^{2^{100}}+1}$$

when $x = 2$ is :

(1) $1-\frac{2^{101}}{4^{101}-1}$ **(2)** $1+\frac{2^{101}}{4^{101}-1}$

(3) $1-\frac{2^{100}}{4^{100}-1}$ **(4)** $1+\frac{2^{101}}{4^{101}-1}$

Q. 12. If $^{20}C_r$ is the coefficient of x^r in the expansion of $(1 + x)^{20}$, then the value of $\sum_{r=0}^{20} r^2\ {}^{20}C_r$ is equal to:

(1) 420×2^{19} **(2)** 420×2^{18}

(3) 380×2^{18} **(4)** 380×2^{19}

Q. 13. A plane P contains the line $x + 2y + 3z + 1 = 0 = x - y - z - 6$, and is perpendicular to the plane $-2x + y + z + 8 = 0$. Then which of the following points lies on P?

(1) (1, 0, 1) **(2)** (2, –1, 1)

(3) (–1, 1, 2) **(4)** (0, 1, 1)

Q. 14. On the ellipse $\frac{x^2}{8}+\frac{y^2}{4} = 1$ let P be a point in the second quadrant such that the tangent at P to the ellipse is perpendicular to the line $x + 2y = 0$. Let S and S′ be the foci of the ellipse and e be its eccentricity. If A is the area of the triangle SPS′ then, the value of $(5 - e^2)$. A is:

(1) 24 **(2)** 6

(3) 14 **(4)** 12

Q. 15. The value of $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right)^{\frac{1}{2}} dx$ is:

(1) $\log_e 4$ **(2)** $\log_e 16$

(3) $4\log_e\left(3+2\sqrt{2}\right)$ **(4)** $2\log_e 16$

Q. 16. The equation $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$ represents a circle with:

(1) centre at (0, –1) and radius $\sqrt{2}$

(2) centre at (0, 1) and radius 2

(3) centre at (0, 1) and radius $\sqrt{2}$

(4) centre at (0, 0) and radius $\sqrt{2}$

Q. 17. If a line along a chord of the circle $4x^2 + 4y^2 + 120x + 675 = 0$, passes through the point (–30, 0) and is tangent to the parabola $y^2 = 30x$, then the length of this chord is :

(1) 5 **(2)** $3\sqrt{5}$

(3) 7 **(4)** $5\sqrt{3}$

Q. 18. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$. If $\vec{c}$ is a vector such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}.\vec{c}=3$, then $\vec{a}.(\vec{b}\times\vec{c})$ is equal to:

(1) –2 **(2)** 6

(3) 2 **(4)** –6

Q. 19. If $A=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}$, $B=\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}$, $i=\sqrt{-1}$ and $Q = A^TBA$, then the inverse of the matrix $A\,Q^{2021}\,A^T$ is equal to:

(1) $\begin{pmatrix}\frac{1}{\sqrt{5}} & -2021\\ 2021 & \frac{1}{\sqrt{5}}\end{pmatrix}$ **(2)** $\begin{pmatrix}1 & 0\\ 2021\,i & 1\end{pmatrix}$

(3) $\begin{pmatrix}1 & 0\\ -2021\,i & 1\end{pmatrix}$ **(4)** $\begin{pmatrix}1 & -2021\,i\\ 0 & 1\end{pmatrix}$

Q. 20. Out of all the patients in a hospital 89% are found to be suffering from heart ailment and 98% are suffering from lungs infection. If K% of them are suffering from both ailments, then K can not belong to the set

(1) {80, 83, 86, 89} **(2)** {79, 81, 83, 85}

(3) {84, 87, 90, 93} **(4)** {84, 86, 88, 90}

Section B

Q. 21. The sum of all integral values of k ($k \neq 0$) for which the equation $\frac{2}{x-1}-\frac{1}{x-2}=\frac{2}{k}$ in x has no real roots, is ______.

Q. 22. The locus of a point, which moves such that the sum of squares of its distances from the points (0, 0), (1, 0), (0, 1), (1, 1) is 18 units, is a circle of diameter d. Then d^2 is equal to ______.

Q. 23. If $y = y(x)$ is an implicit function of x such that $\log_e(x + y) = 4xy$, then $\frac{d^2y}{dx^2}$ at $x = 0$ is equal to _______.

Q. 24. The area of the region

$S = \{(x, y): 3x^2 \leq 4y \leq 6x + 24\}$ is ________.

Q. 25. Let $a, b \in R$, $b \neq 0$. Define a function

$$F(x)=\begin{cases} a\sin\frac{\pi}{2}(x-1), & \text{for } x\leq 0\\ \frac{\tan 2x-\sin 2x}{bx^3}, & \text{for } x>0\end{cases}$$

If f is continuous at $x = 0$, then $10 - ab$ is equal to _________.

Q. 26. If ${}^1P_1+2\cdot{}^2P_2+3\cdot{}^3P_3+.......15\cdot{}^{15}P_{15}={}^qP_r-s$, $0\leq s\leq 1$, then ${}^{q+s}C_{r-s}$ is equal to _________.

Q. 27. A wire of length 36 m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum, and the circumference of the circle is k (meter), then $\left(\frac{4}{\pi}+1\right)k$ is equal to _____.

Q. 28. Let the line L be the projection of the line :

$\frac{x-1}{2}=\frac{y-3}{1}=\frac{z-4}{2}$ in the plane $x - 2y - z = 3$. If d is the distance of the point (0, 0, 6) from L, then d^2 is equal to______.

Q. 29. Let $z = \frac{1-i\sqrt{3}}{2}, i = \sqrt{-1}$. Then the value of

$$21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 + \ldots\ldots + \left(z^{21} + \frac{1}{z^{21}}\right)^3$$

is ______________.

Q. 30. The number of three-digit even numbers, formed by the digits 0, 1, 3, 4, 6, 7 if the repetition of digits is not allowed, is _______.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	1	Geometric Progression	Sequences and Series
2	1	Equation of Line	Straight Line
3	4	System of Linear Equation	Matrices and Determinants
4	1	Solution to Trigonometric Equation	Trigonometric Function
5	4	Addition Theorem	Probability
6	4	Logical Operation	Mathematical Reasoning
7	4	Limit of Sum	Definite Integration
8	3	Linear Differential Equation	Differential Equation
9	3	Mean and Standard Deviation	Statistics
10	2	Differentiation of ITF	Method of Differentiation
11	1	Sum of the Series	Sequences and Series
12	2	Binomial Coefficient	Binomial Theorem
13	4	Plane's Equation	Three Dimensional Geometry
14	2	Tangent	Ellipse
15	2	Definite Integrals	Definite Integration
16	3	Argument	Complex Numbers and Quadratic Equation
17	2	Tangent	Parabola
18	1	Triple Product	Vector Algebra
19	3	Inverse and Transpose	Matrices and Determinants
20	2	Operation on Sets	Sets
21	66	Roots of Equation	Complex Numbers and Quadratic Equation
22	16	Locus of Point	Conic Section
23	40	Derivatives	Method of Differentiation
24	27	Area of the Region	Application of Integrals
25	14	Continuity of Function	Continuity and Differentiability
26	136	Permutations	Permutations and Combinations
27	36	Maxima and Minima	Application of Derivative
28	26	Line	Three Dimensional Geometry
29	13	Cube Roots of Unity	Complex Numbers and Quadratic Equation
30	52	Permutations	Permutations and Combinations

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ANSWERS WITH EXPLANATIONS

Mathematics

Section A

1. Option (1) is correct.

Given $\frac{a}{1-r} = 15$ ...(i)

and $\frac{a^2}{1-r^2} = 150 \Rightarrow \left(\frac{a}{1+r}\right)\left(\frac{a}{1-r}\right) = 150$

Using equation (i),

$\Rightarrow \quad \frac{a}{1+r} = 10$ ...(ii)

Solving equation (i) & (ii), we get

$\frac{15}{1+r} = \frac{10}{1-r}$

$\Rightarrow \quad 15 - 15r = 10 + 10r$

$\Rightarrow \quad 5 = 25r$

$\Rightarrow \quad r = \frac{1}{5}$

From equation (i),

$a = 15(1-r)$

$= 15\left(1-\frac{1}{5}\right)$

$= 15\times\frac{4}{5}$

$= 12$

Now, $ar^2 + ar^4 + ar^6 + \ldots + \infty$

$\Rightarrow \quad S_\infty = \frac{ar^2}{1-r^2} = \frac{12\times\frac{1}{25}}{1-\frac{1}{25}} = \frac{1}{2}$

Hints:

(i) Sum of an infinite G.P $= \frac{a}{1-r}$, where a = first term and r = common difference.

(ii) Sum of square an infinite G.P $= \frac{a^2}{1-r^2}$

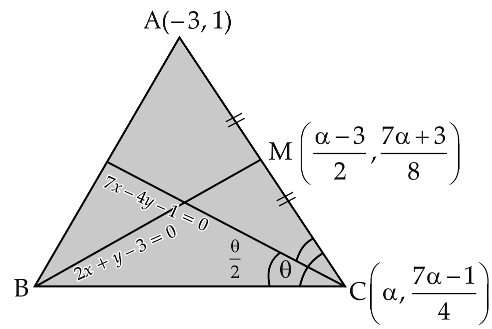
Shortcut Method:

$\frac{15}{1+r} = \frac{10}{1-r}$

This gives r, then find a

Substitute the value of a and r in $\frac{ar^2}{(1-r^2)}$ to get the answer.

2. Option (1) is correct.



Let C $\left(\alpha, \frac{7\alpha-1}{4}\right)$ because C lies on $7x-4y-1=0$

Mid point of AC = M$\left(\frac{\alpha-3}{2}, \frac{7\alpha+3}{8}\right)$

Now, M lies on $2x + y - 3 = 0$,

$2\left(\frac{\alpha-3}{2}\right)+\left(\frac{7\alpha+3}{8}\right)-3 = 0$

$\Rightarrow \quad \alpha = 3$ and $\frac{7\alpha-1}{4} = \frac{7(3)-1}{4} = 5$

So, $\quad C = (3, 5)$

Now, A = (– 3, 1) and C = (3, 5)

Slope of AC, $\frac{y_2-y_1}{x_2-x_1} = \frac{5-1}{3+3} = \frac{4}{6} = \frac{2}{3}$

and slope of $7x - 4y - 1 = 0$ is $\frac{7}{4}$ $\left[\because y = \frac{7}{4}x - 1\right]$

$\therefore \tan\frac{\theta}{2} = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$

$= \frac{\frac{7}{4} - \frac{2}{3}}{1 + \frac{7}{4}\left(\frac{2}{3}\right)} = \frac{1}{2}$

So, $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

$= \frac{2\left(\frac{1}{2}\right)}{1 - \frac{1}{4}} = \frac{4}{3} \Rightarrow \tan\theta = \frac{4}{3}$

Hints:

(i) Mid point of (x_1, y_1) and (x_2, y_2)

$\left[\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}\right]$

(ii) Slope of line joining

(x_1, y_1) and $(x_2, y_2) = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

(iii) $\tan\frac{\theta}{2} = \left[\frac{(m_2 - m_1)}{(1 + m_2 m_1)}\right]$

3. **Option (4) is correct.**

For non-trivial solution

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 3\theta \end{vmatrix} = 0$$

Apply $R_3 \to R_3 - R_2$, we get

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 3\theta \\ 0 & -1 & 1 \end{vmatrix} = 0$$

Apply, $C_2 \to C_2 + C_3$, we get

$$\Rightarrow \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta + 4\sin 3\theta & 4\sin 3\theta \\ \cos^2\theta & 1+\sin^2\theta + 4\sin 3\theta & 4\sin 3\theta \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(1 + 2\sin 3\theta) = 0 \Rightarrow 2\sin 3\theta = -1$

$\Rightarrow \quad \sin 3\theta = -\frac{1}{2}$

$\Rightarrow 3\theta \in \left(0, \frac{3\pi}{2}\right)$ as $\theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad \sin 3\theta = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right)$

[$\because$ sin is -ve, lies in third quadrant]

$\Rightarrow \quad 3\theta = \frac{7\pi}{6}$

$\Rightarrow \quad \theta = \frac{7\pi}{18}$

Hints:

(i) For non trivial solution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

Shortcut Method :

(i) Solve determinant which is equal to 0

(ii) Find θ

4. **Option (1) is correct.**

$\frac{\cos x}{1 + \sin x} = |\tan 2x|, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}, \frac{-\pi}{4}\right\}$

Case-I: $0 \le x < \frac{\pi}{4}$ & $\frac{-\pi}{2} < x < \frac{-\pi}{4}$

$\Rightarrow \frac{\cos x}{1 + \sin x} = \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}$

As $\tan 2x > 0$ & $\tan 2x = \frac{\sin 2x}{1 - 2\sin^2 x}$

$\Rightarrow \quad \cos x(-4\sin^2 x - 2\sin x + 1) = 0$

$\Rightarrow \cos x = 0$ & $4\sin^2 x + 2\sin x - 1 = 0$

$\therefore \sin x = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$,

$\cos x \neq 0$ as $x \neq \pm\frac{\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{10}, \frac{-3\pi}{10}$

Case-II: $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ & $x \in \left(\frac{-\pi}{4} < x < 0\right)$

$\Rightarrow \frac{\cos x}{1 + \sin x} = \frac{-2\sin x \cos x}{\cos^2 x - \sin^2 x}$, as $\tan 2x < 0$

$\Rightarrow \cos x\,(1 + 2\sin x) = 0$

$\Rightarrow \cos x = 0$ & $\sin x = \frac{-1}{2} \Rightarrow x = \frac{-\pi}{6}$

Sum of solutions = $\frac{\pi}{10} - \frac{3\pi}{10} - \frac{\pi}{6} = \frac{-11\pi}{30}$

5. Option (4) is correct.

Probability (Exactly one of A & B) = $\frac{5}{9}$

$\Rightarrow \quad P(A) + P(B) - 2P(A)P(B) = \frac{5}{9}$

$\Rightarrow \quad p + 2p - 4p^2 = \frac{5}{9}$

$\Rightarrow \quad 36p^2 - 27p + 5 = 0$

$\Rightarrow \quad (12p - 5)(3p - 1) = 0$

$\Rightarrow \quad p = \frac{1}{3}$ or $\frac{5}{12}$

$\therefore \quad p_{\max} = \frac{5}{12}$

6. Option (4) is correct.

p	q	r	$p \vee q$	$q \to r$	$\sim r$	$(p \vee q) \wedge (q \to r) \wedge \sim r$	$p \wedge q$
T	T	T	T	T	F	F	T
T	T	F	T	F	T	F	T
T	F	T	T	T	F	F	F
T	F	F	T	T	T	T	F
F	T	T	T	T	F	F	F
F	T	F	T	F	T	F	F
F	F	T	F	T	F	F	F
F	F	F	F	T	T	F	F

$(p \vee q) \wedge (q \to r) \wedge \sim r \to p \wedge q$
T
T
T
F
T
T
T
T

7. Option (4) is correct.

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2}$$

$$\Rightarrow I = \lim_{n\to\infty} \sum_{r=0}^{2n-1} \frac{1}{n} \frac{1}{1 + 4\left(\frac{r}{n}\right)^2}$$

Put, $\frac{r}{n} \to x, \frac{1}{n} \to dx, \lim_{n\to\infty} \frac{0}{n} = 0, \lim_{n\to\infty} \frac{2n-1}{n} = 2$

$$I = \int_0^2 \frac{1}{1+4x^2} dx$$

$$\Rightarrow I = \frac{1}{4} \int_0^2 \frac{dx}{\frac{1}{4} + x^2}$$

$$= \frac{1}{4} \cdot 2\left[\tan^{-1} 2x\right]_0^2 = \frac{1}{2} \tan^{-1}(4)$$

8. Option (3) is correct.

$(y + 1) \tan^2 x\, dx + \tan x\, dy + y\, dx = 0$

So, $\frac{dy}{dx} + (1 + y)\tan x = -y \cot x$

$\Rightarrow \frac{dy}{dx} + y(\tan x + \cot x) = -\tan x$

IF = $e^{\int (\tan x + \cot x)dx} = e^{\int \frac{\tan^2 x + 1}{\tan x} dx} = \tan x$

$\therefore\ y \tan x = \int -\tan^2 x\, dx + C$

$\Rightarrow y \tan x = \int (1 - \sec^2 x) dx + C$

$\Rightarrow y \tan x = x - \tan x + C$

Now $\lim_{x\to 0^+} xy = 1$

$\Rightarrow \lim_{x\to 0^+} \left(\frac{x}{\tan x}\right)(x - \tan x + C) = 1$

$\Rightarrow 1(0 - 0 + C) = 1 \Rightarrow C = 1$

Then, the function $y \tan x = x - \tan x + 1$ at $x = \frac{\pi}{4}$

$\Rightarrow y\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \tan\frac{\pi}{4} + 1$

$\therefore\ y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

9. Option (3) is correct.

$x_1 + x_2 + x_3 + \ldots + x_{19} + 25 = 200$

$\Rightarrow \quad x_1 + x_2 + x_3 + \ldots + x_{19} = 175$

New mean $(\alpha) = \frac{x_1 + x_2 + x_3 + \ldots + x_{19} + 35}{20}$

$= \frac{175 + 35}{20} = 10.5$

$\therefore$ S.D. $= \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + (25)^2}{20} - (10)^2}$

$\Rightarrow \quad 2.5 = \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + (25)^2}{20} - (10)^2}$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 = 1500$

New S.D. $(\sqrt{\beta})$

$= \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + (35)^2}{20} - (10.5)^2}$

$= \sqrt{\dfrac{1500 + (35)^2}{20} - (110.25)}$

$\sqrt{\beta} = \sqrt{26}$

$\therefore (\alpha, \beta) = (10.5, 26)$

10. Option (2) is correct.

Put $x = \sin^2\theta, 0 < x < 1$

$\Rightarrow \sin\theta = \sqrt{x}$

Now, $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\dfrac{1-\sin^2\theta}{\sin^2\theta}}\right)\right)$

$\Rightarrow f(x) = \cos(2\tan^{-1}(\sin\theta))$

$\Rightarrow f(x) = \cos(2\tan^{-1}\sqrt{x}) = \cos\tan^{-1}\left(\dfrac{2\sqrt{x}}{1-x}\right)$

$\Rightarrow f(x) = \dfrac{1-x}{1+x}$

$\Rightarrow f'(x) = \dfrac{(1+x)(-1)-(1-x)\cdot 1}{(1+x)^2} = \dfrac{-2}{(1+x)^2}$

$\Rightarrow f'(x).\ (1-x)^2 = -2\left(\dfrac{1-x}{1+x}\right)^2$

$\Rightarrow (1-x)^2 f'(x) + 2(f(x))^2 = 0$

11. Option (1) is correct.

Adding and subtracting $\dfrac{1}{1-x}$

$\therefore \quad S = \dfrac{1}{1-x} - \dfrac{1}{1-x} + \dfrac{1}{x+1} + \dfrac{2}{x^2+1} + \dfrac{2^2}{x^4+1} + \ldots + \dfrac{2^{100}}{x^{2^{100}}+1}$

$\therefore \quad S = \dfrac{1}{x-1} + \dfrac{2}{1-x^2} + \dfrac{2}{x^2+1} + \dfrac{2^2}{x^4+1} + \ldots + \dfrac{2^{100}}{x^{2^{100}}+1}$

$\therefore \quad S = \dfrac{1}{x-1} - \dfrac{2^{101}}{x^{2^{101}}-1}$

For $x = 2 \Rightarrow S = 1 - \dfrac{2^{101}}{2^{2^{101}}-1} = 1 - \dfrac{2^{101}}{4^{101}-1}$

12. Option (2) is correct.

$\sum_{r=0}^{20} r^2\ {}^{20}C_r = \sum r(r\cdot {}^{20}C_r) = \sum_{r=1}^{20} r\cdot 20\ {}^{19}C_{r-1}$

$= 20\sum_{r=1}^{20}(r-1+1)\ {}^{19}C_{r-1}$

$= 20\left(\sum_{r=1}^{20}(r-1)\ {}^{19}C_{r-1} + \sum_{r=1}^{20} {}^{19}C_{r-1}\right)$

$= 20\left(19\sum_{r=2}^{20} {}^{18}C_{r-2} + \sum_{r=1}^{20} {}^{19}C_{r-1}\right)$

$= (380)2^{18} + 20.2^{19} = 2^{20}(95 + 10)$

$= (105)2^{20} = 420 \times 2^{18}$

13. Option (4) is correct.

A plane which contains the line

$x + 2y + 3z + 1 = 0 = x - y - z - 6$ is

$(x + 2y + 3z + 1) + k(x - y - z - 6) = 0$

$\Rightarrow (1 + k)x + (2 - k)y + (3 - k)z + 1 - 6k = 0$

This plane is perpendicular to the plane $-2x + y + z + 8 = 0$ then

$-2(1 + k) + 1(2 - k) + 1(3 - k) = 0 \Rightarrow k = \dfrac{3}{4}$

Equation of plane is $\dfrac{7}{4}x + \dfrac{5}{4}y + \dfrac{9}{4}z = \dfrac{14}{4}$

$\Rightarrow 7x + 5y + 9z = 14$ ...(i)

For point (1, 0, 1)

$7(1) + 5(0) + 9(1) = 16 \neq 14$

For point (2, –1, 1)

$7(2) + 5(-1) + 9(1) = 18 \neq 14$

For point (1, 1, 2)

$7(-1) + 5(1) + 9(2) = 16 \neq 14$

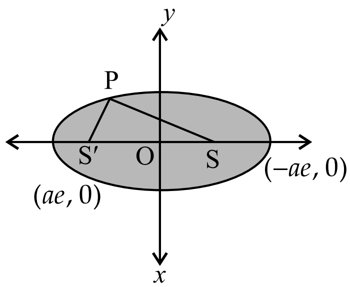
For point (0, 1, 1)

$7(0) + 5(1) + 9(1) = 14$

So, (0, 1, 1) lies on the plane.

14. Option (2) is correct.

Let P($\sqrt{8}$ cos θ, 2sin θ) lies at $\frac{x^2}{8}+\frac{y^2}{4}=1$,



Equation of tangent at point P,

$$\frac{x}{\sqrt{8}}\cos\theta + \frac{y}{2}\sin\theta - 1 = 0$$

Slope = $\frac{-1}{\sqrt{2}}\cot\theta = 2$, because it is perpendicular to $x + 2y = 0$ whose slope is $\frac{-1}{2}$.

$\Rightarrow \cot\theta = -2\sqrt{2} \Rightarrow \cos\theta = \frac{-2\sqrt{2}}{3}$, $\sin\theta = \frac{1}{3}$,

So, point P$\left(\frac{-8}{3},\frac{2}{3}\right)$

$\therefore$ Area (A) $= \frac{1}{2}\times 4\times\frac{2}{3}=\frac{4}{3}$

$e^2 = 1-\frac{4}{8} \Rightarrow e = \frac{1}{\sqrt{2}}$

So, $(5-e^2)\text{A} = \left(5-\frac{1}{2}\right)\times\frac{4}{3} = 6$

15. Option (2) is correct.

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right)^{1/2}dx$$

$$\Rightarrow \int_{1/\sqrt{2}}^{-1/\sqrt{2}}\sqrt{\left[\frac{x-1}{x+1}-\frac{x+1}{x-1}\right]^2}dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{\left(\frac{-4x}{x^2-1}\right)^2}dx$$

$$\Rightarrow \int_{\frac{-1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\sqrt{\frac{16x^2}{(1-x^2)^2}}dx \Rightarrow \int_{\frac{-1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\frac{4|x|}{(1-x^2)}dx,$$

Since $f(x) = \frac{|x|}{1-x^2}$ is an even function,

So apply $\int_{-a}^{a} f|x|\,dx = 2\int_0^a f(x)dx$, we get

$$I = 8\int_0^{1/\sqrt{2}}\frac{x\,dx}{1-x^2} = \left[-4\ln(1-x^2)\right]_0^{1/\sqrt{2}} = \log_e 16$$

16. Option (3) is correct.

Let $z = x + iy$

So, $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$

$$\Rightarrow \frac{z-1}{z+1} = \left(\frac{x+iy-1}{x+iy+1}\right)\times\left(\frac{(x+1)-iy}{(x+1)-iy}\right)$$

$$= \frac{(x+1)(x-1)-iy(x-1)+iy(x+1)-i^2y^2}{(x+1)^2+y^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2+y^2-1)+i(xy+y-xy+y)}{(x+1)^2+y^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

Given, $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = 1$$

$$x^2+y^2-1 = 2y$$

$$x^2+y^2-2y-1 = 0$$

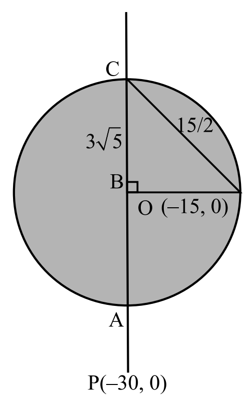
$$(x-0)^2+(y-1)^2 = (\sqrt{2})^2$$

So, centre (0, 1) & Radius = $\sqrt{2}$ Unit

17. Option (2) is correct.

Equation of circle $4x^2 + 4y^2 + 120x + 675 = 0$

Equation of tangent to parabola $y^2 = 30x$ is



$y = mx + \frac{30}{4m}$, (which passes through (–30, 0)

$\therefore\ 0 = -30m + \frac{30}{4m} \Rightarrow 4m^2 = 1 \Rightarrow m = \pm\frac{1}{2}$

For $m = \frac{1}{2} \Rightarrow y = \frac{x}{2} + 15 \Rightarrow x - 2y + 30 = 0$

Length of OB $= \left|\frac{-15+0+30}{\sqrt{5}}\right| = 3\sqrt{5}$, radius of circle $= \frac{15}{2}$

Length of BC $= \sqrt{\frac{225}{4} - 45} \Rightarrow \text{BC} = \frac{3\sqrt{5}}{2}$

Length of chord AC $= 2 \times \frac{3\sqrt{5}}{2} = 3\sqrt{5}$

Similarly, for $m = \frac{-1}{2}$, length of chord AC $= 3\sqrt{5}$

18. Option (1) is correct.

Given, $\vec{a} \times \vec{c} = \vec{b}$

$\Rightarrow -(\vec{c} \times \vec{a}) = \vec{b}$

$\Rightarrow \vec{c} \times \vec{a} = -\vec{b}$

Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c})$

$= [\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{b}\ \vec{c}\ \vec{a}]$

$= \vec{b} \cdot (\vec{c} \times \vec{a})$

$= \vec{b} \cdot (-\vec{b})$

$= -|\vec{b}|^2$

$= -\left(\sqrt{(1)^2 + (-1)^2}\right)^2 \qquad \left[\because |\vec{b}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}\right]$

$= -\left(\sqrt{2}\right)^2$

$= -2$

19. Option (3) is correct.

Given $\quad A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Now, $\quad AA^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \text{...(i)}$

Given $\quad Q = A^TBA$

So, $\quad Q^2 = (A^TBA)(A^TBA) = A^TB^2A$

$\Rightarrow \quad Q^3 = A^TB^3A$

$\Rightarrow \quad Q^{2021} = A^TB^{2021}A$

Now, let $P = AQ^{2021}A^T$

$\Rightarrow \quad P = A(A^TB^{2021}A)A^T$

Since $AA^T = I$ from (i)

$\Rightarrow \quad P = B^{2021}$

Now, $\quad B^2 = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix}$

$B^3 = \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix}$

So, $\quad B^{2021} = \begin{bmatrix} 1 & 0 \\ 2021i & 1 \end{bmatrix}$

Inverse of $P = (P^{-1}) = (B^{2021})^{-1} = \begin{pmatrix} 1 & 0 \\ -2021i & 1 \end{pmatrix}$

20. Option (2) is correct.

Let H & L be the set of people suffering from heart ailment & lungs infections respectively.

$n(H) = 89\%$

$n(L) = 98\%$

Let $n(H \cap L) = K\%$

$\max\{0, n(H) + n(L) - n(H \cup L)\} < n(H \cap L) < \min\{n(H), n(L)\}$

So, $87\% \leq n(H \cap L) \leq 89\%$

Section B

21. Correct answer is [66].

$\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k} \Rightarrow \frac{2x-4-x+1}{(x-1)(x-2)} = \frac{2}{k}$

$\Rightarrow 2x^2 - (6 + k)x + 3k + 4 = 0$
For non real roots D < 0
$\Rightarrow (6 + k)^2 - 8(3k + 4) < 0$
$\Rightarrow k^2 + 12k + 36 - 24k - 32 < 0$
$\Rightarrow (k - 6)^2 - 32 < 0$
Integral value of $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$
Sum of $k = 66$

22. Correct answer is [16].
Let point be $P(x, y)$, then
$x^2 + y^2 + (x - 0)^2 + (y - 1)^2 + (x - 1)^2 + (y - 0)^2 + (x - 1)^2 + (y - 1)^2 = 18$
$\Rightarrow 4x^2 + 4y^2 - 4x - 4y + 4 = 18$

$$\Rightarrow x^2 + y^2 - x - y - \frac{14}{4} = 0$$

$$\therefore \quad \text{Centre} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

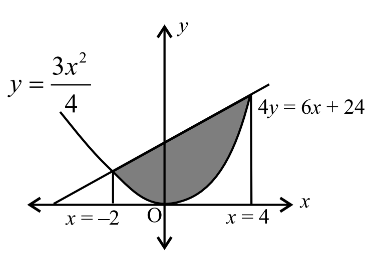
$$r = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{14}{4}} = 2$$

$$\therefore \quad d = 2r = 2(2) = 4$$

So, $d^2 = 16$

23. Correct answer is [40].
When $x = 0$ then $y = 1$
$ln\ (x + y) = 4xy$
$\therefore \quad x + y = e^{4xy}$
Now on differentiating
$\Rightarrow 1 + y' = e^{4xy}(4y + 4xy')$ ...(i)
At (0, 1)$\Rightarrow y'(0) + 1 = 4 \Rightarrow y'(0) = 3$
Now, again on differentiating equation (i)
$\Rightarrow \quad y'' = e^{4xy}(4y + 4xy')^2 + e^{4xy}(4y' + 4y' + 4xy'')$
Now, $y''|_{\text{at }(0, 1)}$
$\Rightarrow y''(0) = 1(4 \times 1 + 0)^2 + 1(4 \times 3 + 4 \times 3 + 0)$
$\Rightarrow y''(0) = 16 + 24 = 40$
$\Rightarrow y''(0) = 40$

24. Correct answer is [27].
$S = \{(x, y) : 3x^2 \leq 4y \leq 6x + 24\}$



Here, $y = \frac{3x^2}{4}$, is the parabala and $4y = 6x + 24$ is the line.
Area of the region

$$A = \int_{-2}^{4}\left(\frac{6x + 24}{4} - \frac{3x^2}{4}\right)dx = \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

$$\Rightarrow A = \frac{3}{4}(16 - 4) + 6(4 + 2) - \frac{1}{4}(64 + 8)$$

$\Rightarrow A = 9 + 36 - 18 \quad \Rightarrow \quad A = 27$ square unit

25. Correct answer is [14].
If function is continuous at $x = 0$
Then, LHL = RHL = $f(0)$
LHL= $-a$

$$\text{RHL} = \lim_{x \to 0} \frac{\tan 2x - \sin 2x}{bx^3}$$

Since, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots.$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$\therefore \text{RHL} = \lim_{x \to 0^+} \frac{\frac{(2x)^3}{3} + \frac{(2x)^3}{6}}{bx^3} = \frac{\frac{8}{3} + \frac{8}{6}}{b} = \frac{4}{b}$$

$$\therefore \quad f(0) = a \sin\left[\frac{\pi}{2}(0 - 1)\right] = a \sin\left(\frac{-\pi}{2}\right) = -a$$

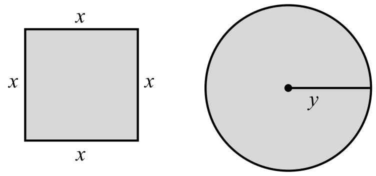
$$\Rightarrow \text{LHL} = \text{RHL} = f(0) \Rightarrow \frac{4}{b} = -a$$

$\Rightarrow \quad -ab = 4 \Rightarrow 10 - ab = 14$

26. Correct answer is [136].
$^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \ldots. + 15 \cdot {}^{15}P_{15}$
$= 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots. + 15 \cdot 15!$
$= 1! + ((3 - 1)\,2!) + ((4 - 1)\,3!) + ((16 - 1)15!)$
$= 1! + 3! - 2! + 4! - 3! + 5! - 4! + \ldots. + 16! - 15!$
$= 16! + 1 - 2!$
$= 16! - 1! = {}^qP_r - s \Rightarrow q = r = 16\ \&\ s = 1$

$$\text{So, } {}^{q+s}C_{r-s} = {}^{17}C_{15} = \frac{17 \times 16}{2} = 136$$

27. Correct answer is [36].



$4x + 2\pi y = 36$ (given) ...(1)

Area $= x^2 + \pi y^2$

$A = x^2 + \frac{1}{\pi}(18 - 2x)^2$ ......using equation (i)

Differentiating w.r.t. x

$$\frac{dA}{dx} = 2x + \frac{2}{\pi}(18 - 2x)(-2)$$

$$\frac{dA}{dx} = \frac{2\pi x + (36 - 4x)(-2)}{\pi}$$

For maxima and minima,

$$\frac{dA}{dx} = 0 \Rightarrow 2\pi x - 72 + 8x = 0$$

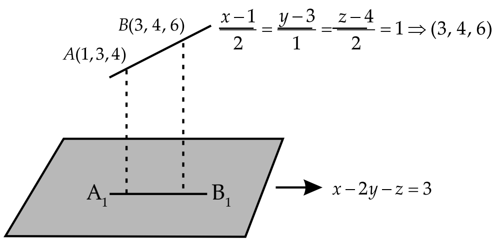
$$\Rightarrow \quad x = \frac{36}{\pi + 4} \Rightarrow \frac{d^2A}{dx^2} > 0$$

$$\Rightarrow \quad y = \frac{18}{\pi + 4} \text{ ......using equation (i)}$$

k [circumference of circle] $= 2\pi y \Rightarrow k = \frac{36\pi}{\pi + 4}$

$\therefore$ Value of $\left(\frac{4 + \pi}{\pi}\right) \times k = 36$

28. Correct answer is [26].



$A_1, B_1 \Rightarrow$ foot of $\perp$ A, B

$$\frac{\alpha - 1}{1} = \frac{\beta - 3}{-2} = \frac{\gamma - 4}{-1} = \frac{-(1 - 6 - 4 - 3)}{6} = 2 \Rightarrow A_1$$

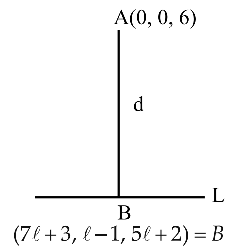
$= (3, -1, 2)$

$$\frac{\alpha - 3}{1} = \frac{\beta - 4}{-2} = \frac{\gamma - 6}{-1} = \frac{-(3 - 8 - 6 - 3)}{6} = \frac{7}{3}$$

$$\Rightarrow B_1 = \left(\frac{16}{3}, \frac{-2}{3}, \frac{11}{3}\right)$$

Drs of A_1B_1 : $\frac{16}{3} - 3, \frac{-2}{3} + 1, \frac{11}{3} - 2 \Rightarrow \frac{7}{3}, \frac{1}{3}, \frac{5}{3}$

$$\therefore L : \frac{x - 3}{\frac{7}{3}} = \frac{y + 1}{\frac{1}{3}} = \frac{z - 2}{\frac{5}{3}} = \ell$$



$$\left(\frac{7}{3}\ell + 3, \frac{\ell}{3} - 1, \frac{5}{3}\ell + 2\right) = B$$

Drs of AB: $\frac{7}{3}\ell + 3, \frac{\ell}{3} - 1, \frac{5}{3}\ell - 4$

$$\overline{AB} \perp L \Rightarrow \frac{7}{3}\left(\frac{7}{3}\ell + 3\right) + \frac{1}{3}\left(\frac{\ell}{3} - 1\right) + \frac{5}{3}\left(\frac{5}{3}\ell - 4\right) = 0$$

$$\left(\frac{49}{9} + \frac{1}{9} + \frac{25}{9}\right)l + \frac{21}{3} - \frac{1}{3} - \frac{20}{3} = 0$$

$\frac{75}{9}\ell = 0 \Rightarrow \ell = 0$. So the point B = (3, –1, 2)

$\therefore$ Distance, AB $= d$

$$= \sqrt{(3 - 0)^2 + (-1 - 0)^2 + (2 - 6)^2}$$

$$\sqrt{9 + 1 + 16} = \sqrt{26}$$

$$\Rightarrow \quad d^2 = 26$$

29. Correct answer is [13].

Given,

$$-z = \frac{-1 + i\sqrt{3}}{2} \text{ and } i = \sqrt{-1}$$

Let $-z = \omega$ or $z = \omega$

Now,

$$z + \frac{1}{z} = -\omega - \frac{1}{\omega} = \frac{-\omega^2 - 1}{\omega} = \frac{\omega}{\omega} = 1 \quad \text{...(i)}$$

And,

$$z^2 + \frac{1}{z^2} = (-\omega)^2 + \frac{1}{(-\omega)^2}$$

$$\omega^2 + \frac{1}{\omega^2} = \frac{\omega^4 + 1}{\omega^2} = \frac{\omega + 1}{\omega^2} = \frac{-\omega^2}{\omega^2} = -1 \qquad ...(ii)$$

Again,

$$z^3 + \frac{1}{z^3} = (-\omega)^3 + \frac{1}{(-\omega)^3} = -1 - 1 = -2 \qquad ...(iii)$$

From equation (i), (ii) and (iii)

$$\left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3$$

$$= 1 - 1 - 8 = -8 \qquad ...(iv)$$

Again,

$$z^6 + \frac{1}{z^6} = (-\omega)^6 + \frac{1}{(-\omega)^6} = (1 + 1)^3 = 8$$

$$\left(z^3 + \frac{1}{z^3}\right)^3 + \left(z^6 + \frac{1}{z^6}\right)^3 = 0 \qquad ...(v)$$

$$\left(z^9 + \frac{1}{z^9}\right)^3 + \left(z^{12} + \frac{1}{z^{12}}\right)^3 = 0 \qquad ...(vi)$$

$$\left(z^{15} + \frac{1}{z^{15}}\right)^3 + \left(z^{18} + \frac{1}{z^{18}}\right)^3 = 0 \qquad ...(vii)$$

$$z^{21} + \frac{1}{z^{21}} = -8 \qquad ...(viii)$$

According to the question, by using above equations.

$\Rightarrow \quad 21 - 8 = 13$

30. Correct answer is [52].

Given digits 0, 1, 3, 4, 6, 7

The number of three-digit even numbers ending with 0

$= 5 \times 4 = 20$

The number of three-digit even numbers ending with 4

$= 4 \times 4 = 16$

The number of three-digit even numbers ending with 6

$= 4 \times 4 = 16$

Total three digits even numbers

$= 20 + 16 + 16 = 52$

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