JEE (Main) MATHEMATICS SOLVED PAPER

2022
29^{th} June Shift 1

Time : 1 Hour **Total Marks : 100**

General Instructions :

1. *In Mathematics Section, there are 30 Questions (Q. no. 1 to 30) having Section A and B.*
2. *Section A consists of 20 multiple choice questions & Section B consists of 10 numerical value type questions. In Section B, candidates have to attempt any five questions out of 10.*
3. *There will be only one correct choice in the given four choices in Section A. For each question for Section A, 4 marks will be awarded for correct choice, 1 mark will be deducted for incorrect choice questions and zero mark will be awarded for not attempted question.*
4. *For Section B questions, 4 marks will be awarded for correct answer and zero for unattempted and incorrect answer.*
5. *Any textual, printed or written material, mobile phones, calculator etc. is not allowed for the students appearing for the test.*
6. *All calculations / written work should be done in the rough sheet is provided with Question Paper.*

Mathematics

Section A

Q. 1. The probability that a randomly chosen 2×2 matrix with all the entries from the set of first 10 primes, is singular, is equal to:

(A) $\frac{133}{10^4}$ **(B)** $\frac{18}{10^3}$

(C) $\frac{19}{10^3}$ **(D)** $\frac{271}{10^4}$

Q. 2. Let the solution curve of the differential equation $x\frac{dy}{dx} - y = \sqrt{y^2 + 16x^2}$, $y(1) = 3$ be $y = y(x)$. Then $y(2)$ is equal to:

(A) 15 **(B)** 11

(C) 13 **(D)** 17

Q. 3. If the mirror image of the point (2,4,7) in the plane $3x - y + 4z = 2$ is (a, b, c), then $2a + b + 2c$ is equal to:

(A) 54 **(B)** 50

(C) –6 **(D)** –42

Q. 4. Let $f: R \to R$ be a function defined by:

$$f(x) = \begin{cases} \max\limits_{t \le x}\{t^3 - 3t\} & ; \quad x \le 2 \\ x^2 + 2x - 6 & ; \quad 2 < x \le 3 \\ [x-3] + 9 & ; \quad 3 < x \le 5 \\ 2x + 1 & ; \quad x > 5 \end{cases}$$

where $[t]$ is the greatest integer less than or equal to t. Let m be the number of points where f is not differentiable and $I = \int_{-2}^{2} f(x)dx$. Then the ordered pair (m, I) is equal to :

(A) $\left(3, \frac{27}{4}\right)$ **(B)** $\left(3, \frac{23}{4}\right)$

(C) $\left(4, \frac{27}{4}\right)$ **(D)** $\left(4, \frac{23}{4}\right)$

Q. 5. Let $\vec{a} = \alpha\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} - \beta\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - 2\hat{k}$ where $\alpha, \beta \in \boldsymbol{R}$, be three vectors. If the projection of a $\vec{a}$ on $\vec{c}$ is $\frac{10}{3}$ and $\vec{b} \times \vec{c} = -6\hat{i} + 10\hat{j} + 7\hat{k}$, then the value of $\alpha + \beta$ is equal to:

(A) 3 **(B)** 4

(C) 5 **(D)** 6

Q. 6. The area enclosed by $y^2 = 8x$ and $y = \sqrt{2x}$ that lies outside the triangle formed by $y = \sqrt{2x}$, $x = 1$, $y = 2\sqrt{2}$, is equal to :

(A) $\frac{16\sqrt{2}}{6}$ **(B)** $\frac{11\sqrt{2}}{6}$

(C) $\frac{13\sqrt{2}}{6}$ **(D)** $\frac{5\sqrt{2}}{6}$

Q. 7. If the system of linear equations

$2x + y - z = 7$

$x - 3y + 2z = 1$

$x + 4y + \delta z = k$, where $\delta, k \in \mathbf{R}$

has infinitely many solutions, then $\delta + k$ is equal to:

(A) – 3 **(B)** 3
(C) 6 **(D)** 9

Q. 8. Let α and β be the roots of the equation $x^2 + (2i - 1) = 0$.

Then, the value of $|\alpha^8 + \beta 8|$ is equal to :

(A) 50 **(B)** 250
(C) 1250 **(D)** 1500

Q. 9. Let $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$ be such that $(p \wedge q)\Delta((p \vee q) \Rightarrow q)$ is a tautology. Then Δ is equal to:

(A) $\wedge$ **(B)** $\vee$
(C) $\Rightarrow$ **(D)** $\Leftrightarrow$

Q. 10. Let $A = [a_{ii}]$ be a square matrix of order 3 such that $a_{ij} = 2^{j-i}$, for all $i, j = 1,2,3$. Then, the matrix $A^2 + A^3 + ... + A^{10}$ is equal to :

(A) $\left(\frac{3^{10}-3}{2}\right)A$ **(B)** $\left(\frac{3^{10}-1}{2}\right)A$

(C) $\left(\frac{3^{10}+1}{2}\right)A$ **(D)** $\left(\frac{3^{10}+3}{2}\right)A$

Q. 11. Let a set $A = A_1 \cup A_2 \cup \ldots \cup A_k$, where $A_i \cap A_j = \phi$ for $i \neq j$, $1 \leq i, j \leq k$. Define the relation R from A to A by $R = \{(x, y): y \in A_i$ if and only if $x \in A_i, 1 \leq i \leq k\}$. Then, R is :

(A) reflexive, symmetric but not transitive
(B) reflexive, transitive but not symmetric
(C) reflexive but not symmetric and transitive
(D) an equivalence relation

Q. 12. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and $a_{n+2} = 2a_{n+1} - a_n + 1$ for all $n \geq 0$.

Then, $\sum_{n=2}^{\infty} \frac{a_n}{7^n}$ is equal to :

(A) $\frac{6}{343}$ **(B)** $\frac{7}{216}$

(C) $\frac{8}{343}$ **(D)** $\frac{49}{216}$

Q. 13. The distance between the two points A and A' which lie on $y = 2$ such that both the line segments AB and $A'B$ (where B is the point (2,3)) subtend angle $\frac{\pi}{4}$ at the origin, is equal to :

(A) 10 **(B)** $\frac{48}{5}$

(C) $\frac{52}{5}$ **(D)** 3

Q. 14. A wire of length 22m is to be cut into two pieces. One of the pieces is to be made into a square and the other into an equilateral triangle. Then, the length of the side of the equilateral triangle, so that the combined area of the square and the equilateral triangle is minimum, is :

(A) $\frac{22}{9+4\sqrt{3}}$ **(B)** $\frac{66}{9+4\sqrt{3}}$

(C) $\frac{22}{4+9\sqrt{3}}$ **(D)** $\frac{66}{4+9\sqrt{3}}$

Q. 15. The domain of the function

$$\cos^{-1}\left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right) \text{ is :}$$

(A) $R - \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

(B) $(-\infty, -1] \cup [1, \infty) \cup \{0\}$

(C) $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right) \cup \{0\}$

(D) $\left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right) \cup \{0\}$

Q. 16. If the constant term in the expansion of $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10}$ is $2^k \cdot l$, where l is an odd integer, then the value of k is equal to:

(A) 6 **(B)** 7
(C) 8 **(D)** 9

Q. 17. $\int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right) dx$

where $[t]$ denotes greatest integer less than or equal to t, is equal to :

(A) –3 **(B)** –2
(C) 2 **(D)** 0

Q. 18. Let PQ be a focal chord of the parabola $y^2 = 4x$ such that it subtends an angle of $\frac{\pi}{2}$ at the point (3,0). Let the line segment PQ be also a focal chord of the ellipse

E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a^2>b^2$. If e is the eccentricity of the ellipse E, then the value of $\frac{1}{e^2}$ is equal to:

(A) $1+\sqrt{2}$ **(B)** $3+2\sqrt{2}$

(C) $1 \quad 2\sqrt{3}$ **(D)** $4+5\sqrt{3}$

Q. 19. Let the tangent to the circle $C_1 : x^2 + y^2 = 2$ at the point $M(-1,1)$ intersect the circle $C_2 : (x-3)^2 + (y-2)^2 = 5$, at two distinct points A and B. If the tangents to C_2 at the points A and B intersect at N, then the area of the triangle ANB is equal to:

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$

(C) $\frac{1}{6}$ **(D)** $\frac{5}{3}$

Q. 20. Let the mean and the variance of 5 observations x_1, x_2, x_3, x_4, x_5 be $\frac{24}{5}$ and $\frac{194}{25}$ respectively. If the mean and variance of the first 4 observation are $\frac{7}{2}$ and a respectively, then $(4a + x_5)$ is equal to:

(A) 13 (B) 15

(C) 17 (D) 18

Section B

Q. 21. Let $S = \{z \in C: |z-2| \le 1, z(1+i) + \bar{z}(1-i) \le 2\}$. Let $|z-4i|$ attains minimum and maximum values, respectively, at $z_1 \in S$ and $z_2 \in S$. If $5(|z_1|^2 + |z_2|^2) = \alpha + \beta\sqrt{5}$, where α and β are integers, then the value of $\alpha + \beta$ is equal to

Q. 22. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx}+\frac{\sqrt{2}y}{2\cos^4 x-\cos 2x} = xe^{\tan^{-1}(\sqrt{2}\cos t 2x)}, 0 < x < \pi/2$ with $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{32}$.

If $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{18}e^{-\tan^{-1}(\alpha)}$., then the value of $3\alpha^2$ is equal to

Q. 23. Let d be the distance between the foot of perpendiculars of the points $P(1, 2, -1)$ and $Q(2, -1, 3)$ on the plane $-x + y + z = 1$. Then d^2 is equal to

Q. 24. The number of elements in the set $S = \{\theta \in [-4\pi, 4\pi]: 3\cos^2 2\theta + 6\cos 2\theta - 10\cos^2\theta + 5 = 0\}$ is

Q. 25. The number of solutions of the equation $2\theta - \cos^2\theta + \sqrt{2} = 0$ in $\boldsymbol{R}$ is equal to

Q. 26. $50\tan\left(3\tan^{-1}\left(\frac{1}{2}\right)+2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) + 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}(2\sqrt{2})\right)$ is equal to

Q. 27. Let $c, k \in \boldsymbol{R}$. If $f(x) = (c+1)x^2 + (1-c^2)x + 2k$ and $f(x+y) = f(x) + f(y) - xy$, for all $x, y \in \boldsymbol{R}$. then the value of $|2(f(1) + f(2) + f(3) + ... + f(20))|$ is equal to

Q. 28. Let H: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a > 0$, $b > 0$, be a hyperbola such that the sum of lengths of the transverse and the conjugate axes is $4(2\sqrt{2}+\sqrt{14})$. If the eccentricity H is $\frac{\sqrt{11}}{2}$, then the value of $a^2 + 2b^2$ is equal to

Q. 29. Let $P_1 : \vec{r}\cdot(2\hat{i}+\hat{j}-3\hat{k})=4$ be a plane. Let P_2 be another plane which passes through the points (2, −3,2), (2, −2, −3) and (1, −4,2). If the direction ratios of the line of intersection of P_1 and P_2 be 16, α, β, then the value of $\alpha + \beta$ is equal to

Q. 30. Let $b_1\, b_2\, b_3\, b_4$ be a 4-element permutation with $b_i \in \{1,2,3, \ldots \ldots ,100\}$ for $1 \le i \le 4$ and $b_i \ne b_j$ for $i \ne j$, such that either b_1, b_2, b_3 are consecutive integers or b_2, b_3, b_4 are consecutive integers. Then the number of such permutations $b_1\, b_2\, b_3\, b_4$ is equal to

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Answer Key

Q. No.	Answer	Topic Name	Chapter Name
Section (A)			
1	C	Basics of Probability	Probability
2	A	Homogeneous differential equation	Differential Equations
3	C	Plane and a Point	Three Dimensional Geometry
4	C	Differentiability of a Function	Continuity and Differentiability
5	A	Scalar and Vector Products	Vector Algebra
6	C	Area Bounded by Curves	Area under Curves
7	B	Systems of Linear Equations	Matrices and Determinants
8	A	Modulus and Argument of Complex Numbers	Complex Numbers
9	C	Tautology and Contradiction	Mathematical Reasoning
10	A	Algebra of Matrices	Matrices and Determinants
11	D	Equivalence Relations	Set Theory and Relations
12	B	Basics of Sequence and Series	Sequences and Series
13	C	Pair of Straight Lines	Point and Straight Line
14	B	Maxima and Minima	Application of Derivatives
15	D	Basics of Functions	Functions
16	D	Multinomial Theorem	Binomial Theorem
17	D	Basics of Definite Integrals	Definite Integration
18	B	Basics of Ellipse	Ellipse
19	C	Pair of Tangents and Chord of Circle	Circle
20	B	Measures of Dispersion	Statistics
Section (B)			
21	26	Geometry of Complex Numbers	Complex Numbers
22	2	Linear Differential Equations	Differential Equations
23	26	Plane and a Point	Three Dimensional Geometry
24	32	Trigonometric Equations	Trigonometric Equations and Inequalities
25	1	Trigonometric Equations	Trigonometric Equations and Inequalities
26	29	Properties of Inverse Trigonometric Functions	Inverse Trigonometric Functions
27	3395	Basics of Differentiation	Differential Coefficient
28	88	Basics of Hyperbola	Hyperbola
29	28	Interaction between Planes	Three Dimensional Geometry
30	1	Permutations	Permutation and Combination

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ANSWERS WITH EXPLANATIONS

Mathematics

Section A

1. Option (C) is correct.

Explanation: Let M be a 2 × 2 matrix such that

$$M = \begin{bmatrix} m & n \\ o & p \end{bmatrix} \text{ and}$$

For M to be a singular matrix, $|M| = 0$

$$\Rightarrow \quad mp - on = 0$$

Case 1: All four elements are equal

$$m = n = o = p$$

$$\Rightarrow \quad mp - on = 0$$

So, number of matrices possible = 10

Case 2: When two prime numbers are used

⇒ Either $m = n$ and $o = p$ or $m = o$ and $n = p$

So, number of matrices possible = $^{10}C_2 \times 2! \times 2!$

$$= \frac{10 \times 9}{2} \times 2 \times 2$$

$$= 180$$

So, number of matrices possible = 10 + 180 = 190

And total number of matrices that can be formed = 10 × 10 × 10 × 10 = 10^4

So, required probability = $\frac{190}{10^4} = \frac{19}{10^3}$

Hint: Take two cases : when all elements are equal and when two prime numbers are used to form the matrix.

Shortcut: Number of matrices possible = when all elements are equal + when two prime number are used

$$= 10 + {}^{10}C_2 + 2! \times 2!$$

$$= 190$$

Total number of matrices = 10^4

$$\text{probability} = \frac{190}{10^4} = \frac{19}{10^3}$$

2. Option (A) is correct.

Explanation: Given differential equation is

$$x\frac{dy}{dx} - y = \sqrt{y^2 + 16x^2}$$

$$\Rightarrow \quad x\frac{dy}{dx} = y + \sqrt{y^2 + 16x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{y^2 + 16x^2}}{x} \quad \ldots(1)$$

Let $\quad y = tx$

$$\Rightarrow \quad \frac{dy}{dx} = t + x\frac{dt}{dx} \quad \ldots(2)$$

From equation (1) and (2), we get

$$t + x\frac{dt}{dx} = \frac{y + \sqrt{y^2 + 16x^2}}{x}$$

$$\Rightarrow \quad t + x\frac{dt}{dx} = \frac{tx + \sqrt{t^2x^2 + 16x^2}}{x}$$

$$\Rightarrow \quad t + x\frac{dt}{dx} = t + \sqrt{t^2 + 16}$$

$$\Rightarrow \quad x\frac{dt}{dx} = \sqrt{t^2 + 16}$$

$$\Rightarrow \quad \frac{dt}{\sqrt{t^2 + 16}} = \frac{dx}{x}$$

Integrating both the sides, we get

$$\int \frac{dt}{\sqrt{t^2 + 16}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \text{In}\,[t + \sqrt{t^2 + 16}\,] = \ln x + \ln c$$

$$\Rightarrow \quad t + \sqrt{t^2 + 16} = xc$$

$$\Rightarrow \quad \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 16} = xc$$

$$\Rightarrow \quad \frac{y}{x} + \sqrt{\frac{y^2 + 16x^2}{x}} = xc$$

$$\Rightarrow \quad y + \sqrt{y^2 + 16x^2} = x^2c$$

Now, $\quad y(1) = 3$

$$\Rightarrow \quad 3 + \sqrt{9 + 16} = \text{c}$$

$$\Rightarrow \quad c = 8$$

$$\Rightarrow \quad y + \sqrt{y^2 + 16x^2} = 8x^2$$

So, at $x = 2$,

$$y + \sqrt{y^2+64} = 32$$

$$\Rightarrow \sqrt{y^2+64} = (32 - y)$$

$$\Rightarrow y^2 + 64 = y^2 + (32)^2 - 64y$$

$$\Rightarrow 64 = 960$$

$$\Rightarrow y = 15$$

$$\therefore y(2) = 15$$

3. Option (C) is correct.

Explanation: Given: The mirror image of (2, 4, 7) in the plane $3x - y + 4z = 2$ is (a, b, c)

As we know, the mirror image of the point (x, y, z) in the plane $ax + by + cz + d = 0$ is given by,

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

$$\Rightarrow \frac{a-2}{3} = \frac{b-4}{-1} = \frac{c-7}{4} = \frac{-2(6+(-4)+28-2)}{(3)^2+(-1)^2+(4)^2}$$

$$\Rightarrow \frac{a-2}{3} = \frac{b-4}{-1} = \frac{c-7}{4} = \frac{-2\,(28)}{26}$$

$$\Rightarrow \frac{a-2}{3} = \frac{-28}{13}, \frac{b-4}{1} = \frac{28}{13}, \frac{c-7}{4} = \frac{-28}{13}$$

$$\Rightarrow a = 2-\frac{84}{13}, b = \frac{28}{13}+4, c = 7-\frac{112}{13}$$

$$\Rightarrow a = \frac{-58}{13}, b = \frac{80}{13}, c = \frac{-21}{13}$$

$$\Rightarrow 2a + b + 2c = 2\left(\frac{-58}{13}\right)+\frac{80}{13}+2\left(\frac{-21}{13}\right)$$

$$\Rightarrow 2a + b + 2c = -6$$

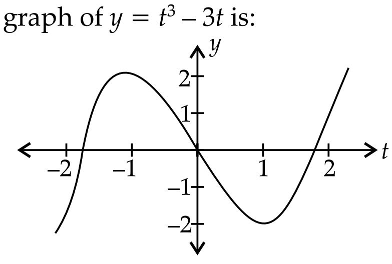
Hint: The mirror image of the point (x, y, z) in the plane $ax + by + cz + d = 0$ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

4. Option (C) is correct.

Explanation: Given: $f : R \to R$

$$f(x) = \begin{cases} \max\limits_{t\le x} \{t^3 - 3t & x \le 2 \\ x^2+2x-6 & 2 < x \le 3 \\ [x-3]+9 & 3 < x \le 5 \\ 2x+1 & x > 5 \end{cases}$$

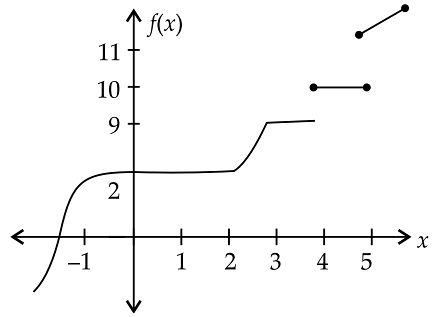
The graph of $y = t^3 - 3t$ is:



For $x \le -1$, $\max\limits_{t<x} \{t^3 - 3t\} = x^3 - 3x$

For $-1 < x \le 2$, $\max\limits_{t<x} \{t^3 - 3t\} = 2$

$$\therefore f(x) = \begin{cases} x^3-3x & x \le -1 \\ 2 & -1 < x \le 2 \\ x^2+2x-6 & 2 < x \le 3 \\ 9 & 3 < x \le 4 \\ 10 & 4 < x < 5 \\ 11 & x = 5 \\ 2x+1 & x > 5 \end{cases}$$



As we know, a function is not differentiable at sharp points and at point of discontinuity

$\Rightarrow f(x)$ is not differentiable at $x = 2, 3, 4, 5$

$\therefore$ The number of points where $f(x)$ is not differentiable = 4

$\Rightarrow m = 4$

Now, $$I = \int_{-2}^{2} f(x)\,dx$$

$$\Rightarrow I = \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$$

$$\Rightarrow I = \int_{-2}^{-1} x^3 - 3x)\,dx + \int_{-1}^{2} 2\,dx$$

$$\Rightarrow I = \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{-2}^{-1} + [2x]_{-1}^{2}$$

$$\Rightarrow I = \left(\frac{1}{4} - \frac{3}{2} - \frac{16}{4} + 6\right) + (4 - (-2))$$

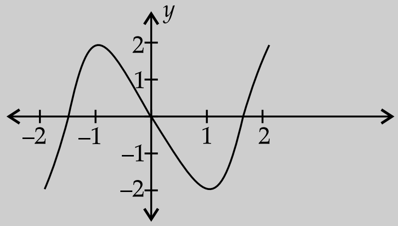
$$\Rightarrow I = \left(\frac{1}{4} - \frac{3}{2} + 2\right) + 6$$

$$\Rightarrow I = \frac{1-6+32}{4}$$

$$\Rightarrow I = \frac{27}{4}$$

$\therefore$ The ordered pair $(m, I) = (4, \frac{27}{4})$

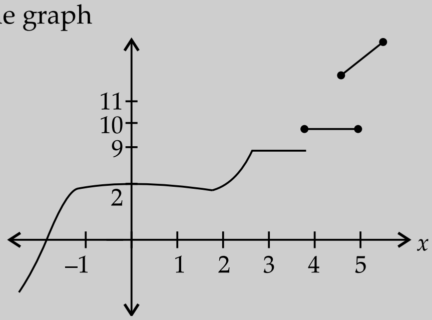
Hint: (i) use the graph of $y = t^3 - 3t$ is



(ii) A function is not differentiable at sharp points and at the point of discontinuity.

Shortcut: $f(x) = \begin{cases} x^3 - 3x & x \le -1 \\ 2 & -1 < x \le 2 \\ x^2 + 2x - 6 & 2 < x \le 3 \\ 9 & 3 < x \le 4 \\ 10 & 4 < x < 5 \\ 11 & x = 5 \\ 2x+1 & x > 5 \end{cases}$

$\Rightarrow f(x)$ is not differentiable at $x = 2, 3, 4, 5$ from the graph



$$\Rightarrow \quad m = 4$$

$$I = \int_{-2}^{2} f(x)\,dx$$

$$\Rightarrow \quad I = \int_{-2}^{-1} x^3 - 3x)\,dx + \int_{-1}^{2} 2\,dx$$

$$\Rightarrow \quad I = \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{-2}^{-1} + [2x]_{-1}^{2}$$

$$\Rightarrow \quad I = \frac{27}{4}$$

$$\therefore \quad (m, I) \equiv (4, \frac{27}{4})$$

5. Option (A) is correct.

Explanation: Given: $\vec{a} = \alpha\hat{i} + 3\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} - \beta\hat{j} + 4\hat{k}$

$\vec{c} = \hat{i} + 2\hat{j} - 2\hat{k}$

The projection of $\vec{a}$ on $\vec{c}$ is $\frac{10}{3}$

As we know, the projection of $\vec{x}$ on $\vec{y}$ is given by $\frac{\vec{x}\cdot\vec{y}}{|\vec{y}|}$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{c}|} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{(\alpha\hat{i} + 3\hat{j} - \hat{k})\,(\hat{i} + 2\hat{j} - 2\hat{k})}{|\ddot{u} + 2\,\hat{}\, - 2\,\hat{}\,|} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{\alpha + 6 + 2}{\sqrt{1+4+4}} = \frac{10}{3}$$

$$\Rightarrow \quad \alpha + 8 = 10$$

$$\Rightarrow \quad \alpha = 2$$

Also, given that $\vec{b} \times \vec{c} = -6\hat{i} + 10\hat{j} + 7\hat{k}$

$$\Rightarrow (3\hat{i} - \beta\hat{j} + 4\hat{k}) \times (\hat{i} + 2\hat{j} - 2\hat{k}) = -6\hat{i} + 10\hat{j} + 7\hat{k}$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -\beta & 4 \\ 1 & 2 & -2 \end{vmatrix} = -6\hat{i} + 10\hat{j} + 7\hat{k}$$

$$\Rightarrow (2\beta - 8)\ \hat{i} - \hat{j}\,(-6-4) + \ \hat{k}\,(6+\beta)\,\hat{k}$$

$$= -6\hat{i} + 10\hat{j} + 7\hat{k}$$

$$\Rightarrow \quad 2\beta - 8 = -6 \text{ or } 6 + \beta = 7$$

$$\Rightarrow \quad \beta = 1$$

$$\therefore \quad \alpha + \beta = 2 + 1 = 3$$

Hint: (i) The projection of $\vec{x}$ on $\vec{y}$ is $\frac{\vec{x}\cdot\vec{y}}{|\vec{y}|}$

(ii) If $\vec{x} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{y} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

then $\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & j & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

Shortcut: The projection of $\vec{a}$ on $\vec{c}$ is $\frac{10}{3}$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{c}|} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{\alpha + 6 + 2}{\sqrt{1^2 + (2)^2 + (-2)^2}} = \frac{10}{3}$$

$$\alpha = 2$$

and $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -\beta & 4 \\ 1 & 2 & -2 \end{vmatrix} = -6\hat{i}+10\hat{j}+7\hat{k}$

$\Rightarrow \quad 2\beta - 8 = -6, 6 + \beta = 7$

$\Rightarrow \quad \beta = 1$

$\Rightarrow \quad \alpha + \beta = 3$

6. Option (C) is correct.

Explanation: Given curves are $y^2 = 8x$ and $y = \sqrt{2}x$

Let us find the intersection point of both the curves:

$$(\sqrt{2}x)^2 = 8x$$

$\Rightarrow \quad 2x^2 = 8x$

$\Rightarrow \quad 2x^2 - 4x = 0$

$\Rightarrow \quad x(x-4) = 0$

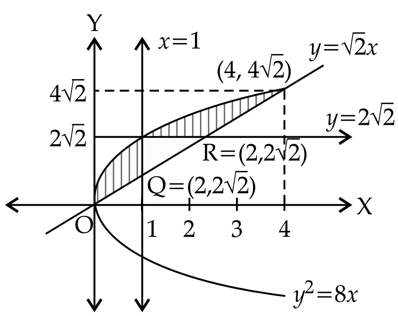
$\Rightarrow \quad x = 0, 4$

$\Rightarrow \quad y = 0, 4\sqrt{2}$

So intersection points are (0, 0) and $(4, 4\sqrt{2})$

Let PQR be the triangle formed by $y = \sqrt{2}x$, $x = 1$ and $y = 2\sqrt{2}$

$\Rightarrow P = (1, 2\sqrt{2}), Q = (1, \sqrt{2}), R = (2, 2\sqrt{2})$



$$\text{Area of } OPSRQ = \int_0^4 (\sqrt{8x} - \sqrt{2}x)\, dx$$

$$= \left[\sqrt{8}\left(\frac{2}{3}x^{3/2}\right) - \frac{\sqrt{2}}{2}x^2\right]_0^4$$

$$= \frac{2\sqrt{8}}{3}(4)^{3/2} - \frac{1}{\sqrt{2}}(4)^2$$

$$= \frac{4\sqrt{2}}{3}(8) - \frac{16}{\sqrt{2}}$$

$$\Rightarrow \quad \text{Area of } OPSRQ = \left(\frac{32\sqrt{2}}{3} - 8\sqrt{2}\right) \text{ sq. units}$$

Now, area of $\Delta PQR = \frac{1}{2} \times (PQ) \times (PR)$

$$= \frac{1}{2} \times (2\sqrt{2} - \sqrt{2}) \times 1$$

$$= \frac{1}{2} \times \sqrt{2}$$

$$\Rightarrow \quad \text{Area of } \Delta PQR = \frac{\sqrt{2}}{2} \text{ sq. units}$$

So, Required area = area of shaded region

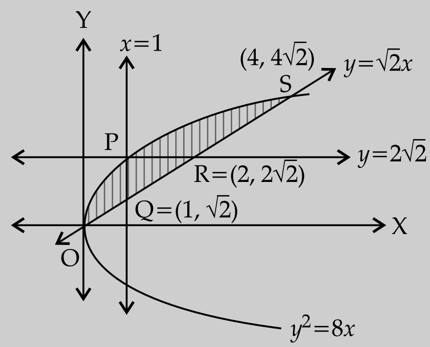
= Area of $OPSRQ$ – Area of ΔPQR

$$= \frac{32\sqrt{2}}{3} - 8\sqrt{2} - \frac{\sqrt{2}}{2}$$

$$= \frac{64\sqrt{2} - 48\sqrt{2} - 3\sqrt{2}}{6}$$

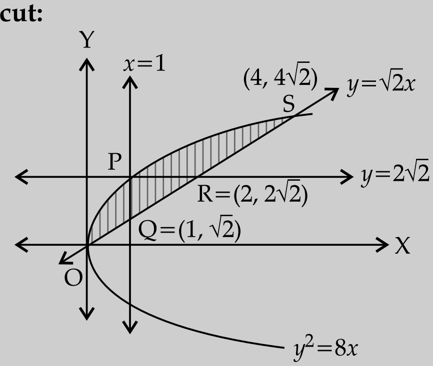
$$= \frac{13\sqrt{2}}{6} \text{ sq. units}$$

Hint:



Required area = Area of $OPSRQ$ – Area of ΔPQR

Shortcut:



Required area = Area of $OPSRQ$ – Area of ΔPQR

$$= \int_0^4 (\sqrt{8x} - \sqrt{2}x) - \frac{1}{2} \times (2\sqrt{2} - \sqrt{2}) \times 1$$

$$= \frac{32\sqrt{2}}{3} - 8\sqrt{2} - \frac{\sqrt{2}}{2}$$

$$= \frac{13\sqrt{2}}{6} \text{ sq. units}$$

7. Option (B) is correct.

Explanation: Given: A system of linear equation has infinitely many solution.

$$2x + y - z = 7$$
$$x - 3y + 2z = 1$$
$$x + 4y + \delta z = k$$

As we know, if a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_3z = d_2$$
$$a_3x + b_2y + c_3z = d_3$$

has infinitely many solutions then, $D = D_1 = D_2 = D_3 = 0$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

So, $D = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & \delta \end{vmatrix} = 0$

$\Rightarrow 2(-3\delta - 8) - 1(\delta - 2) - 1(4 + 3) = 0$

$\Rightarrow -6\delta - 16 - \delta + 2 - 7 = 0$

$\Rightarrow \quad 7\delta = -21$

$\Rightarrow \quad \delta = -3$

Also, $D_3 = \begin{vmatrix} 2 & 1 & 7 \\ 1 & -3 & 1 \\ 1 & 4 & k \end{vmatrix} = 0$

$\Rightarrow 2(-3k - 4) - 1(\text{k} - 1) + 7(4 + 3) = 0$

$\Rightarrow -6k - 8 - k + 1 + 49 = 0$

$\Rightarrow \quad 7k = 42$

$\Rightarrow \quad k = 6$

$\Rightarrow \quad \delta + k = -3 + 6 = 3$

Hint: If a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_3z = d_2$$
$$a_3x + b_2y + c_3z = d_3$$

has infinite solutions, the $D = D_1 = D_2 = D_3 = 0$ where

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Shortcut: If a system of linear equations has infinite solutions, then

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow \quad \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & \delta \end{vmatrix} = 0$

$\Rightarrow \quad \delta = -3$

and $\quad \begin{vmatrix} 2 & 1 & 7 \\ 1 & -3 & 1 \\ 1 & 4 & k \end{vmatrix} = 0$

$\Rightarrow \quad k = 6$

$\therefore \quad \delta + k = 3$

8. Option (A) is correct.

Explanation: Given: α and β are roots of $x^2 + (2i - 1) = 0$

$\Rightarrow \quad x^2 + (2i - 1) = 0$

$\Rightarrow \quad x^2 = 1 - 2i$

$\Rightarrow \quad \alpha^2 = 1 - 2i$ and $\beta^2 = 1 - 2i$

$\Rightarrow \quad \alpha^2 = \beta^2$

$\Rightarrow \quad (\alpha^2)^4 = (\beta^2)^4$

$\Rightarrow \quad \alpha^8 = \beta^8$

$\therefore \quad \alpha^8 + \beta^8 = 2\alpha^8$

$\therefore \quad \alpha^8 + \beta^8 = 2(\alpha^2)^4$

$\Rightarrow \quad \alpha^8 + \beta^8 = 2(1 - 2i)^4$

$\Rightarrow \quad |\alpha^8 + \beta^8| = 2\,|1 - 2i|^4$

$\Rightarrow \quad |\alpha^8 + \beta^8| = 2\left(\sqrt{(1)^2 + (-2)^2}\right)^4$

$\Rightarrow \quad |\alpha^8 + \beta^8| = 2\left(\sqrt{5}\right)^4$

$\Rightarrow \quad |\alpha^8 + \beta^8| = 2(25) = 50$

Hint: (1) The modulus of a complex number $a + bi = \sqrt{a^2 + b^2}$

9. Option (C) is correct.

Explanation:

p	q	$p \vee q$	$(p \vee q) \Rightarrow q$	$p \wedge q$	$(p \wedge q) \wedge (p \vee q) \Rightarrow q)$	$(p \wedge q) \vee ((p \vee q) \Rightarrow q)$	$(p \wedge q) \Rightarrow ((p \vee q) \Rightarrow q)$	$(p \wedge q) \Leftrightarrow ((p \vee q) \Rightarrow q)$
T	T	T	T	T	T	T	T	T
T	F	T	F	F	F	F	T	T
F	T	T	T	F	T	T	T	F
F	F	F	T	F	F	T	T	F

Clearly, $(p \wedge q) \Rightarrow ((p \vee q) \Rightarrow q)$ is a tautology.

Hint: Tautology is a statement which is always true.

Shortcut: $(p \vee q) \Rightarrow q$
$\equiv \sim (p \vee q) \vee q$
$\equiv (\sim p \wedge \sim q) \vee q$
{By De Morgan's law}
$\equiv (\sim p \vee q) \wedge (\sim q \vee q)$
$\equiv (\sim p \vee q) \wedge T$
$\equiv (\sim p \vee q)$

And $(p \wedge q) \Rightarrow ((p \vee q) \Rightarrow q)$
$\equiv (p \wedge q) \Rightarrow (\sim p \vee q)$
$\equiv \sim (p \wedge q) \vee (\sim p \vee q)$
$\equiv T$

So option '*C*' is correct.

10. Option (A) is correct.

Explanation: Given: $a_{ij} = 2^{j-i}$

$$\Rightarrow A = \begin{bmatrix} 2^{1-1} & 2^{2-1} & 2^{3-1} \\ 2^{1-2} & 2^{2-2} & 2^{3-2} \\ 2^{1-3} & 2^{2-3} & 2^{3-3} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

$$\text{Now, } A^2 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 1+1+1 & 2+2+2 & 4+4+4 \\ 1/2+1/2+1/2 & 1+1+1 & 2+2+2 \\ 1/4+1/4+1/4 & 1/2+1/2+1/2 & 1+1+1 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 3 & 6 & 12 \\ 3/2 & 3 & 6 \\ 3/4 & 3/2 & 3 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

$\Rightarrow A^2 = 3A$

Also, $A^3 = A^2.A = 3A.A = 3A^2 = 3(3A) = 3^2A$

Similarly $A^4 = A^3.A = 3^2A.A = 3^2(A^2) = 3^2(3A) = 3^3A$

$\Rightarrow A^2 + A^3 + A^4 + \ldots\ldots + A^9 + A^{10} = 3A + 3^2A + 3^3A + \ldots\ldots + 3^8A + 3^9A$

$= 3A\,(1 + 3 + 3^2 + \ldots\ldots + 3^7 + 3^8)$

$$= 3A\left(\frac{3^{9-1}}{3-1}\right)$$

$$= 3A\left(\frac{3^{9-1}}{2}\right)$$

$$= \left(\frac{3^{10}-3}{2}\right)A$$

Hint: (i) Simplify using multiplication of matrices.

(ii) Sum of *G.P.* with a as first term, r as common ratio and n as number of terms is given by $\dfrac{a\,(r^n-1)}{r-1}$.

Shortcut: $A = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

$\Rightarrow A^2 = 3A$

$A^3 = A^2A = 3^2A$

$\therefore A^2 + A^3 + \ldots\ldots + A^{10} = 3A + 3^2A + \ldots\ldots + 3^9A$

$$= A\left\{\frac{3\,(3^9-1)}{3-1}\right\}$$

$$= \left(\frac{3^{10}-3}{2}\right)A$$

11. Option (D) is correct.

Explanation: Set $A = A_1 \cup A_2 \cup A_3 \ldots \cup A_x$, where $A_i \cap A_j = \phi;\ i \neq j,\ 1 \leq i, j \leq k$

And relation $R = \{(x, y) : y \in A_i : \textit{iff}\ x \in A_i;\ 1 \leq i \leq k\}$

(1) Symmetric: If $(x, y) \in R$, then $(y, x) \in R$

$\therefore$ Given relation is symmetric.

(2) Reflexive: $\because (a, a) \in R$ for all $a \in A_i$

$\therefore$ Given relation is reflexive

(3) Transitive: If $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow y \in A$; *iff* $x \in A_i$ and $z \in A_i$ *iff* $y \in A_i$

$\Rightarrow z \in A$; *iff* $x \Rightarrow A_i$

$\Rightarrow (x, z) \in R$

$\therefore$ Given relation is transitive.

Since, given relation is symmetric, reflexive and transitive

$\therefore$ It is an equivalence relation.

Hint: (i) Recall the definition of symmetric, reflexive and transitive relation.

(ii) A relation is said to be equivalence relation if it is symmetric, reflexive and transitive.

Shortcut: Let $A = =\{1, 2, 3, 4\}$

$\Rightarrow R = \{(1\,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1)$ $(3, 2), (3, 3)\}$

$\because$ R is reflexive transitive and symmetric.

$\therefore$ It is an equivalence relation.

12. Option (B) is correct.

Explanation:

Given, $a_0 = a_1 = 0$ and $a_{n+2} = 2a_{n+1} - a_n + 1$ $\forall\, n \geq 0$

$\Rightarrow \quad a_2 = 2a_0 - a_0 + 1 = 1$

and $\quad a_3 = 2a_2 - a_1 + 1 = 3$

and $\quad a_4 = 2a_3 - a_2 + 1 = 6$

And $\quad a_5 = 2a_4 - a_3 + 1 = 10$

$$\therefore \quad a_n = \frac{n(n-1)}{2}$$

$$\text{Let} \quad p = \sum_{n=2}^{\infty} \frac{a_n}{7^n}$$

$$\Rightarrow \quad p = \sum_{n=2}^{\infty} \frac{n\,(n-1)}{2 \cdot 7^n}$$

$$\Rightarrow \quad p = \frac{1}{7^2} + \frac{3}{7^3} + \frac{6}{7^4} + \frac{10}{7^5} + \ldots\ldots \quad \ldots\text{(i)}$$

$$\Rightarrow \quad \frac{p}{7} = \frac{1}{7^3} + \frac{3}{7^4} + \frac{6}{7^5} + \frac{10}{7^6} + \ldots\ldots \quad \ldots\text{(ii)}$$

Equation (i) – equation (ii), we get

$$\frac{6p}{7} = \frac{1}{7^2} + \frac{2}{7^3} + \frac{3}{7^4} + \frac{4}{7^5} + \ldots\ldots \quad \ldots\text{(iii)}$$

$$\Rightarrow \quad \frac{6p}{7^2} = \frac{1}{7^3} + \frac{2}{7^4} + \frac{3}{7^5} + \frac{4}{7^6} + \ldots\ldots \quad \ldots\text{(iv)}$$

Equation (iii) – equation (iv), we get

$$\frac{6p}{7}.\left(1 - \frac{1}{7}\right) = \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \frac{1}{7^5} + \ldots\ldots$$

$$\Rightarrow \quad \frac{6p}{7}\left(\frac{6}{7}\right) = \frac{\frac{1}{7^2}}{1 - \frac{1}{7}}$$

$$\Rightarrow \quad \left(\frac{6}{7}\right)^2 p = \frac{1}{(7)\,(6)}$$

$$\Rightarrow \quad p = \frac{7}{216}$$

Hint:

(i) Use $an = \frac{n(n-1)}{2}$

(ii) Simplify given sequence and try to convert it into the form of infinite *G.P.* and solve further.

(iii) Sum of infinite *G.P.* with first term a and common ratio r ($r < 1$) is given by $\frac{a}{1-r}$.

13. Option (C) is correct.

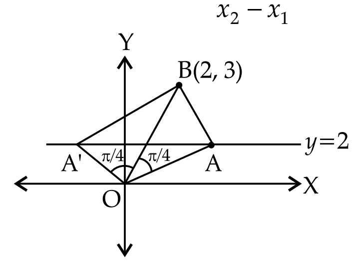
Explanation: Given: A and A' lies on $y = 2$

Let coordinates of $A \equiv (x_1, 2)$ and $A' \equiv (x_2, 2)$

Let slope of OA be m_1 and OB be m_2

As we know slope of line passing through (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$$\Rightarrow \quad m_1 = \frac{2}{x_1} \text{ and } m_2 = \frac{3}{2}$$

Also, we know that if angle between two lines having slope m_1 and m_2 is θ, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \quad \tan\frac{\pi}{4} = \left|\frac{\frac{2}{x_1} - \frac{3}{2}}{1 + \left(\frac{2}{x_1}\right)\left(\frac{3}{2}\right)}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{\frac{2}{x_1} - \frac{3}{2}}{1 + \frac{3}{x_1}}\right|$$

$\Rightarrow \quad 1 = \left|\frac{(4-3x_1)}{2(x_1+3)}\right|$

$\Rightarrow \quad \pm 1 = \frac{4-3x_1}{2(x_1+3)}$

$\Rightarrow \quad 4-3x_1 = \pm(2x_1+6)$

$\Rightarrow \quad 4-3x_1 = 2x_1+6$

and $\quad 4-3x = -2x_1-6$

$\Rightarrow \quad 5x_1 = -2$ and $x_1 = 10$

$\Rightarrow \quad x_1 = \frac{-2}{5}$ and $x_1 = 10$

$\because$ x is positive for A and negative for A'

$\therefore \quad x_1 = 10$ and $x_2 = \frac{-2}{5}$

$\Rightarrow \quad A \equiv (10, 2)$ & $A' \equiv (\frac{-2}{5}, 2)$

$\therefore$ by distance formula,

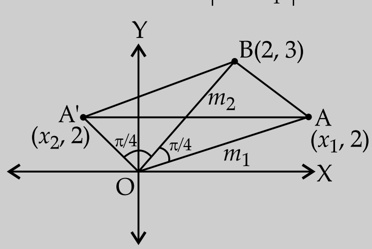
$$AA' = \sqrt{\left(10+\frac{2}{5}\right)^2 + (2-2)^2}$$

$\Rightarrow \quad AA' = \frac{52}{5}$ units

Hints: (1) The angle between two lines having slope m_1 and m_2 is θ and $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

Shortcut: $m_1 = \frac{2}{x_1}$ and $m_2 = \frac{3}{2}$

$\Rightarrow \quad \tan\frac{\pi}{4} = \left|\frac{\frac{2}{x_1}-\frac{3}{2}}{1+\frac{3}{x_1}}\right|$



$\Rightarrow \quad \pm 1 = \frac{4-3x_1}{2(x_1+3)}$

$\Rightarrow \quad x_1 = \frac{-2}{5}$ and $x_1 = 10$

$\therefore \quad A \equiv (10, 2)$ and $A' = \left(\frac{-2}{5}, 2\right)$

$\Rightarrow \quad AA' = \sqrt{\left(10+\frac{2}{5}\right)^2 + (2-2)^2}$

$AA' = \frac{52}{5}$ units

14. Option (B) is correct.

Explanation: Given: A wire of length 22 m

Let length of the side of triangle be x and the length of the side of square be y and p be the length of wire formed into triangle

$\therefore \quad p = 3x$

and $\quad 22-p = 4y$

$\Rightarrow \quad x = \frac{p}{3}$ and $y = \frac{1}{4}(22-p)$

Now, area of triangle $= \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4}\left(\frac{p}{3}\right)^2$

and $\quad$ area of square $= y^2 = \left[\frac{1}{4}(22-p)\right]^2$

$\therefore \quad$ Total area $= \frac{\sqrt{3}}{4}\frac{p^2}{9} + \frac{1}{16}(22-p)^2$

$\Rightarrow \quad A = \frac{\sqrt{3}}{36}p^2 + \frac{1}{16}(22^2 + p^2 - 44p)$

$\Rightarrow \quad A = \frac{\sqrt{3}}{36}p^2 + \frac{p^2}{16} - \frac{22}{8}p + \frac{22^2}{16}$

For A to be minimum, $\frac{dA}{dp} = 0$

$\Rightarrow \quad \frac{dA}{dp} = 2\left(\frac{\sqrt{3}}{36}p^2\right) + 2\left(\frac{p}{16}\right) - \frac{22}{8} = 0$

$\Rightarrow \quad \frac{dA}{dp} = \frac{\sqrt{3}}{18}p + \frac{p}{8} - \frac{22}{8} = 0$

$\Rightarrow \quad p\left(\frac{\sqrt{3}}{18} + \frac{1}{8}\right) = \frac{22}{8}$

$\Rightarrow \quad p = \frac{22/8}{\frac{4\sqrt{3}+9}{72}}$

$\Rightarrow \quad p = \frac{22}{8} \times \frac{72}{4\sqrt{3}+9}$

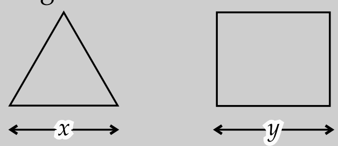
$\Rightarrow \quad x = \frac{p}{3} = \frac{22 \times 9}{(4\sqrt{3}+9)\,3}$

$\Rightarrow \quad x = \frac{66}{4\sqrt{3}+9}$

Hint: (i) The area of equilateral triangle with side a is $\frac{\sqrt{3}}{4}a^2$

(ii) Find total area in terms of length of wire of triangle and find critical points.

Shortcut: Let p be the length of wire formed into triangle.



$$\Rightarrow \quad x = \frac{p}{3} \text{ and } y = \frac{(22-p)}{4}$$

$$\therefore \quad \text{Total area} = \frac{\sqrt{3}}{4}\left(\frac{p^2}{9}\right) + \frac{(22-p)^2}{16}$$

$$\Rightarrow \quad A = \left(\frac{\sqrt{3}}{18} + \frac{1}{16}\right)p^2 + \frac{22^2}{16} - \frac{22}{8}p$$

$$\Rightarrow \quad \frac{dA}{dp} = \left(\frac{\sqrt{3}}{18} + \frac{1}{8}\right)p - \frac{22}{8}$$

For minimum A, $\frac{dA}{dp} = 0$

$$\Rightarrow \quad p\left(\frac{\sqrt{3}}{18} + \frac{1}{8}\right) = \frac{22}{8}$$

$$\Rightarrow \quad p = \frac{22 \times 9}{(4\sqrt{3} + 9)}$$

$$\Rightarrow \quad x = \frac{66}{(4\sqrt{3} + 9)}$$

15. Option (D) is correct.

Explanation: Let $f(x) = \cos^{-1}\left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right)$

As we know the domain of $\cos^{-1} y$ is [–1, 1]

$$\Rightarrow \quad -1 \le \left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right) \le 1$$

$$\Rightarrow \quad -\frac{\pi}{2} \le \sin^{-1}\left(\frac{1}{4x^2-1}\right) \le \frac{\pi}{2}$$

$$\Rightarrow \quad -1 \le \frac{1}{4x^2-1} \le 1$$

So, $\frac{1}{4x^2-1} \ge -1$ and $\frac{1}{4x^2-1} \le 1$

$$\Rightarrow \quad \frac{1}{4x^2-1} + 1 \ge 0 \text{ and } \frac{1}{4x^2-1} - 1 \le 0$$

$$\Rightarrow \quad \frac{4x^2}{4x^2-1} \ge 0 \text{ and } \frac{2-4x^2}{4x^2-1} \le 0$$

$$\Rightarrow \quad \frac{x^2}{(2x-1)(2x+1)} \ge 0$$

and $\frac{(1-\sqrt{2}x)(1+\sqrt{2}x)}{(2x-1)(2x+1)} \le 0$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right) \cup \{0\}$$

and $x \in \left(-\infty, \frac{1}{\sqrt{2}}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right) \cup \{0\}$$

Hint: (i) Domain of $\cos^{-1} y$ is [–1, 1]

(ii) Recall wavy curve method for solving rational inequalities.

Shortcut:

Let $f(x) = \cos^{-1}\left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right)$

For domain of $f(x)$,

$$\Rightarrow \quad -1 \le \left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right) \le 1$$

$$\Rightarrow \quad -1 \le \frac{1}{4x^2-1} \le 1$$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right) \cup \{0\}$$

16. Option (D) is correct.

Explanation: We have to find the constant term in expansion of $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10}$

$$= \left(\frac{3x^8 - 2x^7 + 5}{x^5}\right)^{10}$$

$$= x^{-50}(3x^8 - 2x^7 + 5)^{10}$$

For constant term in the expansion of $x^{-50}(3x^8 - 2x^7 + 5)^{10}$ we will find the coefficient of x^{50} in $(3x^8 - 2x^7 + 5)^{10}$

As we know, the coefficient of x^r in the expansion of $(a + b + c)^n$ is given by $\frac{n!}{r_1!\,r_2!\,r_3!}(a)^{r_1}(b)^{r_2}(c)^{r_3}$

where $r^1 + r^2 + r^3 = n$

So here coefficient of x^{50} in the expansion of $(3x^8 - 2x^7 + 5)^{10}$

$$= \frac{10!}{r_1!\,r_2!\,r_3!}\left(3x^8\right)^{r_1}\left(-2x^7\right)^{r_2}(5)^{r_3}$$

$$= \frac{10!}{r_1!\,r_2!\,r_3!}(3)^{r_1}(-2)^{r_2}(5)^{r_3}(x)^{8r_1+7r_2}$$

Here $r_1 + r_2 + r_3 = 10$ and $8r_1 + 7r_2 = 50$

Let $r_1 = 1 \Rightarrow r_2 = 6$ and $r_3 = 3$

$\therefore$ Coefficient is $\frac{10!}{1!\;6!\;3!}(3)^1(-2)^6(5)^3$

$$= \frac{10\times9\times8\times7}{3\times2}\times3\times(2)^6\times(5)^3$$

$$= 2 \times (5)^4 \times (2)^2 \times 7 \times (2)^6 \times (3)^2$$

$$= (2^9)(3)^2(5)^4(7)$$

Now, $(2^K)\,l = (2^9)(3)^2(5)^4(7)$

$\Rightarrow K = 9$

Hint: $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10} = x^{-50}(3x^8 - 2x^7 + 5)^{10}$.

So find the coefficient of x^{50} in $(3x^8 - 2x^7 + 5)^{10}$

Shortcut: The general term of $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10}$

is $T_{r+1} = \frac{10!}{r_1!\,r_2!\,r_3!}(3)^{r_1}(-2)^{r_2}(5)^{r_3}(x)^{3r_1+2r_2-5r_3}$

Where $r_1 + r_2 + r_3 = 10$

and $3r_1 + 2r_2 - 5r_3 = 0$ (for constant term)

Solving above two equations, we get,

$r_1 = 1, r_2 = 6, r_3 = 3$

$\therefore$ Constant term $= \frac{10!}{6!\;3!}(3)^1(-2)^6(5)^3$

$= 2^9(3)^2(5)^4(7)$

$\therefore$ $k = 9$.

17. Option (D) is correct.

Explanation: Let $I = \int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right)dx$

Now,

$$\cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right) = \cos\left(\pi x - \pi\left[\frac{x}{2}\right]\right)$$

$$\cos\left(\pi x - \pi\left[\frac{x}{2}\right]\right) = \begin{cases}\cos(\pi x - 0) & 0 < x < 2\\ \cos(\pi x - \pi) & 2 \le x < 4\\ \cos(\pi x - 2\pi) & 4 \le x < 5\end{cases}$$

$$\therefore I = \int_0^2 \cos(\pi x)dx + \int_2^4 \cos(\pi x - \pi)dx + \int_4^5 \cos(\pi x - 2\pi)dx$$

$$\Rightarrow I = \left[\frac{\sin(\pi x)}{\pi}\right]_0^2 + \left[\frac{\sin(\pi x - \pi)}{\pi}\right]_2^4 + \left[\frac{\sin(\pi x - 2\pi)}{\pi}\right]_4^5$$

$$\Rightarrow I = \frac{1}{\pi}(\sin 2\pi - \sin 0 + \sin(4\pi - \pi)) - \sin(2\pi - \pi) + \sin(5\pi - 2\pi) - \sin(4\pi - 2\pi)$$

$$\Rightarrow I = \frac{1}{\pi}(0)$$

$$\Rightarrow I = 0$$

Hint: Break the integral into 3 parts, $x \in [0, 2)$ and $x \in [2, 4)$ and $x \in [4, 5)$ and solve it.

Shortcut: Let $I = \int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right)dx$

$\Rightarrow I =$

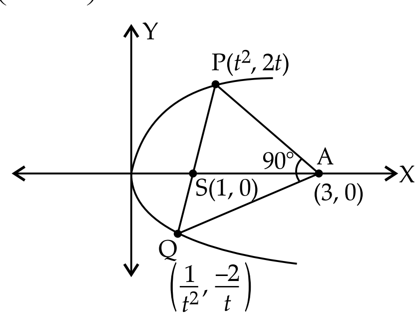
$$\int_0^2 \cos(\pi x)dx + \int_2^4 \cos(\pi x - \pi)dx + \int_4^5 \cos(\pi x - 2\pi)dx$$

$\Rightarrow I = 0$

18. Option (B) is correct.

Explanation: Given: Focal chord of $y^2 = 4x$ is PQ and it subtends an angle of $\frac{\pi}{2}$ at point (3, 0)

Let parametric coordinates of point P be $(t^2, 2t)$ then parametric coordinates of point Q be $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$



$\because \quad AP \perp AQ$

$\therefore \quad (m_{AP})(m_{AQ}) = -1$

$$\Rightarrow \left(\frac{2t}{t^2-3}\right)\left(\frac{\frac{-2}{t}}{\frac{1}{t^2}-3}\right) = -1$$

$$\Rightarrow \frac{-4t^2}{(t^2-3)(1-3t^2)} = -1$$

$\Rightarrow \quad 4t^2 = -3t^4 + 10t^2 - 3$

$\Rightarrow \quad 3t^4 - 6t^2 + 3 = 0$

$\Rightarrow \quad (t^2 - 1)^2 = 0$

$\Rightarrow \quad t = 1$

$\therefore$ Coordinates of point P and Q are (1, 2) and (1, –2) respectively.

Since, line segment PQ is also a focal chord of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a^2 > b^2$.

$\therefore$ P and Q must be end points of latus rectum

$\Rightarrow \frac{2b^2}{a} = 4$ and $ae = 1$

As we know, $\quad b^2 = a^2(1-e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2$

$\Rightarrow \quad b^2 = a^2 - 1$

$$\Rightarrow \frac{a^2-1}{a} = 2$$

$\Rightarrow \quad a^2 - 2a - 1 = 0$

$\Rightarrow \quad a = 1 + \sqrt{2}$

$$\Rightarrow e = \frac{1}{a} = \frac{1}{1+\sqrt{2}}$$

$$\Rightarrow e^2 = \frac{1}{3+2\sqrt{2}}$$

$$\Rightarrow \frac{1}{e^2} = 3+2\sqrt{2}$$

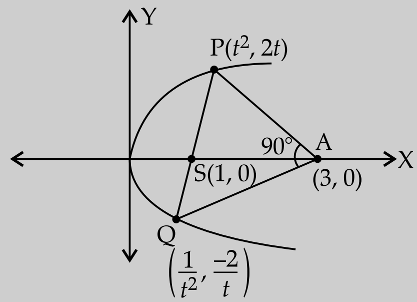
Hint:

(i) Coordinates of end point of any focal chord of parabola $y^2 = 4ax$ are $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(ii) P and Q must be end points of latus rectum of ellipse.

(iii) The end points of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a^2 > b^2$ are $\left(ae, \frac{b^2}{a}\right)$ and $\left(ae, \frac{-b^2}{a}\right)$

Shortcut:



$\therefore AP \perp AQ$

$\therefore (m_{AP})(m_{AQ}) = -1$

$$\Rightarrow \left(\frac{2t}{t^2-3}\right)\left(\frac{\frac{-2}{t}}{\frac{1}{t^2}-3}\right) = -1$$

$\Rightarrow (t^2 - 1)^2 = 0$

$\Rightarrow t = 1$

So, coordinates of point P and Q are (1, 2) and (1, –2) respectively

$\therefore$ P and Q must be end point of latus rectum.

$\Rightarrow \frac{2b^2}{a} = 4$ and ae = 1

$\Rightarrow \frac{b^2}{a} = 2$ and $a^2e^2 = a^2 - b^2 = 1$

$\Rightarrow a = 1 + \sqrt{2}$

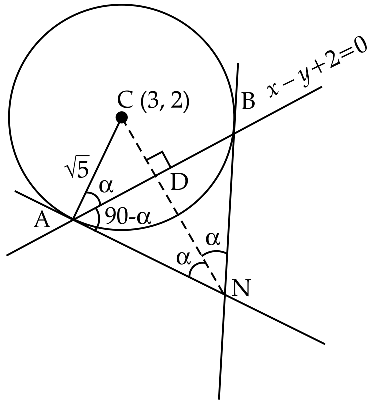
$\Rightarrow \frac{1}{e^2} = 3+2\sqrt{2}$

19. Option (C) is correct.

Explanation: Circles $C_1 : x^2 + y^2 = 2$ and $C_2 : (x-3)^2 + (y-2)^2 = 5$

Now, equation of tangent to the circle C_1 and M (–1, 1) is given by $x(-1) + y(1) = 2$

$\Rightarrow \quad x - y + 2 = 0$



Let $\quad \angle ANB = 2\alpha$

$\therefore \quad \angle CAD = \alpha$

Now, $\quad CD = \left|\dfrac{3-2+2}{\sqrt{1^2+1^2}}\right| = \dfrac{3}{\sqrt{2}}$

Apply Pythagoras' theorem in ΔACD, we get

$$(CD)^2 + (AD)^2 = (AC)^2$$

$$\Rightarrow \quad AD = \sqrt{5-\frac{9}{2}}$$

$$\Rightarrow \quad AD = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \tan\alpha = \frac{CD}{AD} = 3$$

$$\Rightarrow \quad \sin\alpha = \frac{CD}{AC} = \frac{3}{\sqrt{10}}$$

Now, in ΔADN,

$$\sin\alpha = \frac{AD}{AN} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \quad AN = \frac{\sqrt{10}}{3}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{5}}{3}$$

Now, area of $\Delta ANB = \dfrac{1}{2}(AN)^2 \sin 2\alpha$

$$= \frac{1}{2}\left(\frac{5}{9}\right)(2\sin\alpha\cos\alpha)$$

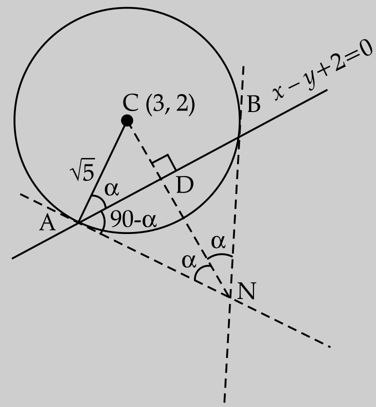
$$= \frac{5}{9}\times\frac{3}{\sqrt{10}}\times\frac{1}{\sqrt{10}}$$

$$= \frac{1}{6} \text{ square unit}$$

Hint: (i) Equation of the tangent to the circle $x^2 + y^2 = r^2$ at point (x_1, y_1) is given by $xx_1 + yy_1 = r^2$

(ii) Draw the diagram as per given question and solve further using the concert of circle.

Shortcut:



Let $\quad \angle ANB = 2\alpha$

$\therefore \quad \angle CAD = \alpha$

Now, $\quad CD = \left|\dfrac{3-2+2}{\sqrt{1^2+1^2}}\right| = \dfrac{3}{\sqrt{2}}$

And $\quad AD = \sqrt{(AC)^2 - (AD)^2}$

$$= \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\alpha = \frac{CD}{AC} = \frac{3}{\sqrt{10}}$$

Now, in ΔADN, $\sin\alpha = \dfrac{AD}{AN}$

$$\Rightarrow \quad AN = \frac{\sqrt{5}}{3}$$

Now, area of $\Delta ANB = \dfrac{1}{2}(AN)^2 \sin 2\alpha$

$$= \frac{1}{2}\left(\frac{5}{9}\right)\left(2.\frac{3}{\sqrt{10}}.\frac{1}{\sqrt{10}}\right)$$

$$= \frac{1}{6} \text{ square unit}$$

20. Option (B) is correct.

Explanation: The observations are x_1, x_2, x_3, x_4,

x_5 mean of $x_1, x_2, x_3, x_4, x_5 = \frac{24}{5}$

$\Rightarrow \quad \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{24}{5}$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + x_5 = 24 \quad ...(i)$

and mean of $\quad x_1, x_2, x_3, x_4 = \frac{7}{2}$

$\Rightarrow \quad \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{7}{2}$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 14 \quad ...(2)$

From eq. (1) and (2)

$x_5 = 24 - 14 = 10$

variance of $x_1, x_2, x_3, x_4, x_5 = \frac{194}{25}$

$\Rightarrow \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}{5} = \frac{194}{25} + \left(\frac{24}{5}\right)^2$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \frac{194}{25} + \frac{576}{5} = 154$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = 154 - (10)^2$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = 54$

variance of $x_1, x_2, x_3, x_4 = a$

$\Rightarrow \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{4} - \frac{49}{4} = a$

$\Rightarrow \frac{54}{4} - \frac{49}{4} = a$

$\Rightarrow \quad a = \frac{5}{4}$

$\Rightarrow \quad 4a + x_5 = 5 + 10 = 15$

Hint : Mean $= \frac{\Sigma x}{n}$

Variance $= \frac{\Sigma x^2}{n} - \frac{\Sigma x}{n}$

Shortcut : $\frac{\Sigma x_1}{5} = \frac{24}{5}$

$\Rightarrow \quad \Sigma x_1 = 24$

$\sigma_1^2 = \frac{\Sigma x_1^2}{5} - \left(\frac{24}{5}\right)^2 = \frac{194}{25}$

$\Rightarrow \quad \Sigma x_1^2 = 154$

and $x_1 + x_2 + x_3 + x_4 = 14$

$\Rightarrow \quad x_5 = 10$

and $\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{4} - \frac{49}{4} = a$

$\Rightarrow \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4a + 49$

$\Rightarrow \quad x_5^2 = 154 - 4a - 49$

$\Rightarrow \quad 100 = 154 - 4a - 49$

$\Rightarrow \quad 4a = 5$

$\Rightarrow \quad 4a + x_5 = 15$

Section B

21. Correct answer is [26].

Explanation: Given: $S = \{Z \in C : |z - 2| \leq 1, z(1 + i) + \bar{z}(1 - i) \leq 2\}$

Let $\quad z = x + iy$

Now, $\quad |z - 2| \leq 1$

$\Rightarrow \quad |(x - 2) + iy| \leq 1$

$\Rightarrow \quad (x - 2)^2 + y^2 \leq 1$

$\Rightarrow$ It represents the region inside circle whose centre is (2, 0) and radius is 1.

Now, $\quad z(1 + i) + \bar{z}(1 - i) \leq 2$

$\Rightarrow \quad (x + iy)(1 + i) + (x - iy)(1 - i) \leq 2$

$\Rightarrow \quad x - y - 1 \leq 0$

$\Rightarrow$ It represents the all points which lies on and above the line $x - y - 1 = 0$

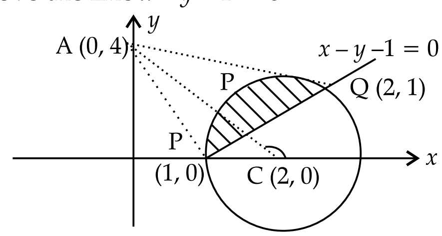


Fig.

Now, $|z - 4i|$ represents distance of a point $A(0, 4)$ from z.

Now, $AP = \sqrt{17}$ and $AQ = \sqrt{13}$

$\therefore \; |z - 4i|_{max} = AP$ and $|z - 4i|_{min} = AD$

Let coordinates of point D be $(\cos\theta + 2, \sin\theta)$

Now, $(m)_{AC} = \tan\theta = -2$

$\Rightarrow \cos\theta = -\frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$

$\therefore$ coordinates of point D is $\left(2 - \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

So, $\quad z_1 = 2 - \frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}$ and $z_2 = 1$

Now, $\quad |z_1| = \sqrt{\left(2 - \frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2}$

$\Rightarrow \quad |z_1| = \sqrt{4 + \frac{1}{5} - \frac{4}{\sqrt{5}} + \frac{4}{5}}$

$\Rightarrow \quad |z_1| = \sqrt{\frac{5\sqrt{5} - 4}{5}}$

$\Rightarrow \quad |z_1|^2 = \frac{5\sqrt{5} - 4}{\sqrt{5}}$

Now, $5(|z_1|^2 + |z_2|^2) = 5\left(\frac{5\sqrt{5}-4}{\sqrt{5}}+1\right)$

$= 30 - 4\sqrt{5}$

$\Rightarrow \alpha + \beta\sqrt{5} = 30 - 4\sqrt{5}$

$\Rightarrow \quad \alpha = 30, \beta = -4$

$\therefore \quad \alpha + \beta = 26$

Hint :

(i) $|z-2| \leq 1$, represents the region inside the circle whose centre is (2, 0) and radius is 1.

(ii) $z(1+i) + \bar{z}(1-i) \leq 2$, represents all the points which lies on and above the line $x - y - 1 = 0$

(iii) Find the required region using above points and solve further.

22. Option (B) is correct.

Explanation: Given: $\frac{dy}{dx} + \frac{\sqrt{2}}{2\cos^4 x - \cos 2x}$

$y = x\, e^{[\tan^{-1}(\sqrt{5}\cot 2x)]}$; $x \in \left(0, \frac{\pi}{2}\right)$

It is linear differential equation.

Comparing above differential equation with

$\frac{dy}{dx} + py = Q,$

we get, $\quad p = \frac{\sqrt{2}}{2\cos^4 x - \cos 2x}$

and $\quad Q = xe^{\tan^{-1}(\sqrt{5}\cot 2x)}$

Now, $\quad I.F. = e$

So, $\quad \int p.dx = \int \frac{\sqrt{2}}{2\cos^4 x - \cos 2x}dx$

$\Rightarrow \int p.dx = \int \frac{\sqrt{2}}{\frac{1}{2}(2\cos^4 x)^2 - \cos 2x}dx$

$\Rightarrow \int p.dx = \int \frac{\sqrt{2}}{\frac{1}{2}(1+\cos 2x)^2 - \cos 2x}dx$

$\Rightarrow \int p.dx = \int \frac{2\sqrt{2}}{1+\cos^2 2x}dx$

$\Rightarrow \int p.dx = \int \frac{2\sqrt{2}\sec^2 2x}{2+\tan^2 2x}dx$

Let $t = \tan 2x \Rightarrow dt = 2\sec^2 2x\, dx$

$\Rightarrow \int p.dx = \sqrt{2}\int \frac{dt}{(\sqrt{2}) + t^2}$

$\Rightarrow \int p.dx = \sqrt{2}, \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)$

$\Rightarrow \int p.dx = \tan^{-1}\left(\frac{\tan 2x}{\sqrt{2}}\right)$

$\therefore IF = e$

So, solution of the given differential equation is given by $y\,(IF) = \int Q.(I.F.).dx$

$\Rightarrow y\, e^{\tan^{-1}\left(\frac{\tan 2x}{\sqrt{2}}\right)} = \int x\, e^{\tan -1(\sqrt{2}\cot 2x)}.$

$e^{\tan -1(\frac{\tan 2x}{\sqrt{2}})}\, dx \quad$ ...(i)

Now, $\tan^{-1}(\sqrt{2}\cot 2x) + \tan^{-1}\left(\frac{\tan 2x}{\sqrt{2}}\right)$

$= \tan^{-1}\left(\frac{\sqrt{2}\cot 2x + \tan 2x/\sqrt{2}}{1-1}\right) = \frac{\pi}{2}$

From equation (i),

$y\, e^{\tan -1\left(\frac{\tan 2x}{\sqrt{2}}\right)} = \int e^{\pi/2} x\, dx$

$\Rightarrow \quad y\, e^{\tan -1\left(\frac{\tan 2x}{\sqrt{2}}\right)} = e^{\frac{\pi}{2}}.\frac{x^2}{2} + C \quad$ ...(ii)

$\because \quad y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{32}$

$\therefore \quad \frac{\pi^2}{32}e^{\frac{\pi}{2}} = e^{\frac{\pi}{2}}.\frac{\pi^2}{32} + C$

$\Rightarrow \quad C = 0$

put $x = \frac{\pi}{3}$ in equation (ii), we get

$y\left(\frac{\pi}{3}\right).e^{\tan^{-1}\left(-\sqrt{\frac{3}{2}}\right)} = e^{\frac{\pi}{2}}.\frac{\pi^2}{18}$

$\Rightarrow \frac{\pi^2}{18}e^{\tan^{-1}(\alpha)}.e^{\tan -1\left(-\sqrt{\frac{3}{2}}\right)} = e^{\frac{\pi}{2}}.\frac{\pi^2}{18}$

$\Rightarrow \quad e^{\tan^{-1}(-\alpha) + \tan^{-1}\left(-\sqrt{\frac{3}{2}}\right)} = e^{\frac{\pi}{2}}$

$\Rightarrow \quad \tan^{-1}(-\alpha) + \tan^{-1}\left(-\sqrt{\frac{3}{2}}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \alpha\sqrt{\frac{3}{2}} = 1$$

$$\Rightarrow \quad \alpha^2 = \frac{2}{3}$$

$$\Rightarrow \quad 3\alpha^2 = 2$$

Hint :

(i) Solution of linear differential equation $\frac{dy}{dx} + py = Q$, where p and Q are the function of x is given by $y\ (I.F.) = \int Q.(\text{I.F.})\,dx$

where $I.F. = e^{\int p.dx}$

(ii) Use $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(iii) Use $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

23. Correct answer is [26].

Explanation: Given: Equation of plane is $-x + y + z = 1$

Points $p(1, 2, -1)$ and $Q(2, -1, 3)$ lie on same side of the plane.

Now, perpendicular distance of a point p from plane $-x + y + z - 1 = 0$ is

$$d_1 = \left|\frac{-1+2-1-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}$$

And perpendicular distance of a point Q from plane $-x + y + z\ 1 = 0$ is

$$d_2 = \left|\frac{-2-1+3-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}$$

$$\because \quad d_1 = d_2$$

$\therefore\ \overrightarrow{PQ}$ is parallel to given plane.

So, distance between P and Q = distance between their foot of perpendiculars

$$\Rightarrow \quad |\overrightarrow{PQ}| = \sqrt{(2-1)^2+(-1-2)^2+(3+1)^2}$$

$$= \sqrt{26}$$

$$\Rightarrow \quad d = \sqrt{26}$$

$$\Rightarrow \quad d^2 = 26$$

Hint :

(i) Perpendicular distance of a point P and Q from the plane is same.

(ii) Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Shortcut : Points P and Q lie on same side of plane. Perpendicular distance of a point P and Q from the plane $-x + y + z = 1$ is same. So, distance between the foot of perpendiculars = PQ

$$\Rightarrow \quad d = \sqrt{(2-1)^2+(-1-2)^2+(3+1)^2}$$

$$= \sqrt{26}$$

$$\Rightarrow \quad d^2 = 26$$

24. Correct answer is [32].

Explanation: Given: $3\cos^2 2\theta + 6\cos 2\theta - 10\cos^2 \theta + 5 = 0;\ \theta \in [-4\pi, 4\pi]$

$\Rightarrow 3\cos^2 2\theta + 6\cos 2\theta - 5(1 + \cos 2\theta) + 5 = 0$

$\{\because 2\cos^2\theta = 1 + \cos 2\theta\}$

$$\Rightarrow \quad 3\cos^2 2\theta + \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta(3\cos 2\theta + 1) = 0$$

$$\Rightarrow \quad \cos 2\theta = 0 \text{ or } \cos 2\theta = -\frac{1}{3}$$

Case-1 If $\cos 2\theta = 0$

$$\Rightarrow \quad 2\theta = 2n + 1)\frac{\pi}{2};\ n \in I$$

$$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{4}$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{4}, \pm 3\frac{\pi}{4}, \pm 5\frac{\pi}{4}, \ldots\ldots\ldots \pm 15\frac{\pi}{4}$$

$\therefore$ For $\theta \in [-4\pi, 4\pi]$, 16 values of θ is possible for this case.

Case-2 If $\cos 2\theta = -\frac{1}{3}$

Let $\quad \cos\alpha = -\frac{1}{3}$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(-\frac{1}{3}\right);\ \alpha \in \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \quad 2\theta = 2n\pi \pm \alpha;\ \alpha \in \left(\frac{\pi}{2}, \pi\right)$$

$\Rightarrow \qquad \theta = n\pi \pm \dfrac{\alpha}{2}$

$\therefore$ For $\theta \in [-4\pi, 4\pi]$, 16 values of θ is possible for this case.

So, number of elements in the set S is 32.

> **Hint :**
> **(i)** Simplify given trigonometric equation using $1 + \cos 2\theta = 2\cos^2\theta$ and solve further.
> **(ii)** General solution of $\cos x = 0$ is $x = (2n+1)\dfrac{\pi}{2};\ n \in I.$
> **(iii)** General solution of $\cos x = \cos \alpha$ is $x = 2n\pi \pm \alpha;\ \alpha \in (0, \pi)$

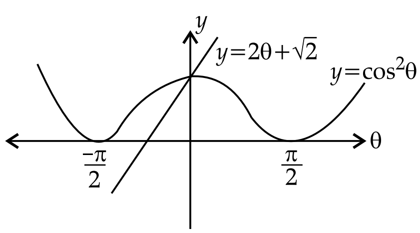
> **Shortcut :** $3\cos^2 2\theta + 6\cos 2\theta - 10\cos^2\theta + 5 = 0$
>
> $\Rightarrow \quad 3\cos^2 2\theta + \cos 2\theta = 0$
>
> $\Rightarrow \quad \cos 2\theta = 0$ or $\cos 2\theta = -\dfrac{1}{3}$
>
> If $\quad \cos 2\theta = 0$
>
> $\Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{4},\ n \in I$
>
> $\therefore$ For $\theta \in [-4\pi, 4\pi]$, 16 values of θ is possible.
>
> If $\cos 2\theta = -\dfrac{1}{3}$
>
> Similarly, 16 values of θ is possible for $\theta \in [-4\pi, 4\pi]$ for this case.
>
> $\therefore$ Total solution = 32 for $\theta \in [-4\pi, 4\pi]$

25. Correct answer is [1].

Explanation: Given: $2\theta - \cos^2\theta\ \sqrt{2} = 0$

$\Rightarrow \cos^2\theta = 2\theta + \sqrt{2}$

Lets draw the graph of $y = \cos^2\theta$ and $y = 2\theta + \sqrt{2}$



$\because$ Both graphs intersect at one point.

$\therefore$ Number of solution for given equation is 1.

> **Hint :**
> **(i)** Draw the graph of $y = \cos^2\theta$ and $y = 2\theta + \sqrt{2}$ and find intersection point of both graph.

> **Shortcut :** $2\theta - \cos^2\theta + \sqrt{2} = 0$
>
> $\Rightarrow \quad \cos^2\theta = 2\theta + \sqrt{2}$
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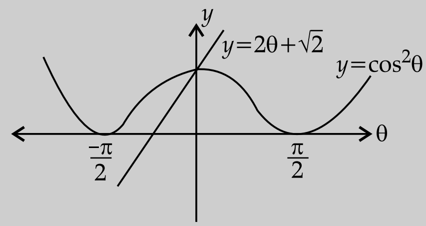
> 
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>
> Since, both graphs intersect at one point
>
> So, Number of solution for given equation is 1.

26. Correct answer is [29].

Explanation: Let $A = 50\tan\left(3\tan^{-1}\left(\dfrac{1}{2}\right) + 2\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right)\right) + 4\sqrt{2}\tan\left(\dfrac{1}{2}\tan^{-1}(2\sqrt{2})\right)$

Let $\quad B = \tan\left(\dfrac{1}{2}\tan^{-1}(2\sqrt{2})\right)$

Let $\quad 2\theta = \tan^{-1}(2\sqrt{2});\ 2\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore \quad \tan 2\theta = 2\sqrt{2}$

$\Rightarrow \quad \dfrac{2\tan\theta}{1-\tan^2\theta} = 2\sqrt{2} \qquad$ ...(i)

$\Rightarrow \quad 2\sqrt{2}\tan^2\theta + 2\tan\theta - 2\sqrt{2} = 0$

$\Rightarrow \quad 2\sqrt{2}\tan^2\theta + 4\tan\theta - 2\tan\theta - 2\sqrt{2} = 0$

$\Rightarrow \quad (\tan\theta + \sqrt{2})(2\sqrt{2}\tan\theta - 2) = 0$

$\Rightarrow \quad \tan\theta = -\sqrt{2}$ or $\dfrac{1}{\sqrt{2}}$

$\because \quad \theta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

$\therefore \quad \tan\theta = -\sqrt{2}$ is not possible

So, $\quad \tan\theta = \dfrac{1}{\sqrt{2}}$

$\therefore \quad B = \dfrac{1}{\sqrt{2}}$

Now, $\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) = \tan^{-1}(2)$

Let $C = \tan\left(3\tan^{-1}\left(\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$

$\Rightarrow C = \tan$

$$\left(\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}(2)\right)$$

$$\Rightarrow C = \tan\left(\tan^{-1}\left(\frac{1}{2}\right) + 2\left[\tan^{-1}\left(\frac{\frac{1}{2}+2}{1-1}\right)\right]\right)$$

$$\Rightarrow C = \tan\left(\tan^{-1}\left(\frac{1}{2}\right) + 2\left(\frac{\pi}{2}\right)\right)$$

$$\Rightarrow C = \tan\left(\pi + \tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow C = \tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow C = \frac{1}{2}$$

$$\therefore A = 50\left(\frac{1}{2}\right) + 4\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow A = 25 + 4 = 29$$

Hint :

(i) Convert $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ in terms of $\tan^{-1}$ and simplify further using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) Assume $2\theta = \tan^{-1}\left(2\sqrt{2}\right)$ and solve further.

Shortcut : Let $A = 50$

$$\tan\left(3\tan^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$$

$$+ 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}(2\sqrt{2})\right)$$

Let $B = \tan\left(\frac{1}{2}\tan^{-1}(2\sqrt{2})\right)$ and

$$\left(3\tan^{-1}\left(\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$$

Let $2\theta = \tan^{-1}\left(2\sqrt{2}\right); 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \tan 2\theta = 2\sqrt{2}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$$

$$\therefore B = \frac{1}{\sqrt{2}}$$

Now, $C = \tan\left(3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}(2)\right)$

$$\Rightarrow C = \tan\left(\tan^{-1}\left(\frac{1}{2}\right) + \pi\right)$$

$$\Rightarrow C = \frac{1}{2}$$

$$\therefore A = 50\left(\frac{1}{2}\right) + 4\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow A = 29$$

27. Correct answer is [3395].

Explanation:

$$f(x) = (c+1)x^2 + (1-c^2)x + 2k \quad ...(1)$$

and $f(x+y) = f(x) + f(y) - xy \ \forall x, y \in R$

$$\lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0}\frac{f(y)-xy}{y}$$

$$\Rightarrow f(x) = f(0) - x$$

$$f(x) = -\frac{1}{2}x^2 + f(0).\,x + \lambda$$

but $f(0) = 0 \Rightarrow \lambda = 0$

$$f(x) = -\frac{1}{2}x^2 + (1-c).x \quad ...(2)$$

as $f = 1 - c^2$

Comparing equation (1) and (2)

We obtain, $c = -\frac{3}{2}$

$$\therefore f(x) = -\frac{1}{2}x^2 - \frac{5}{4}x$$

$$\left|2\Sigma_{x=1}^{20} f(x)\right| = \Sigma_{x=1}^{20} x^2 + \frac{5}{2}.\Sigma_{x=1}^{20} x$$

$$= 2870 + 525 = 3395$$

28. Correct answer is [88].

Explanation: Given: Equation of hyperbola is

$$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; a > 0, b > 0$$

And eccentricity of H is $e = \frac{\sqrt{11}}{2}$

And sum of length of transverse and conjugate axis is $2a + 2b = 4\left(2\sqrt{2} + \sqrt{14}\right)$

As we know, $e = \sqrt{1 + \frac{b^2}{a^2}}$

$\Rightarrow \quad e^2 = 1 + \frac{b^2}{a^2}$

$\Rightarrow \quad \frac{11}{4} = 1 + \frac{b^2}{a^2}$

$\Rightarrow \quad b^2 = \frac{7}{4}a^2$

$\Rightarrow \quad b = \frac{\sqrt{7}}{2}a$

$\because \quad 2a + 2b = 4\left(2\sqrt{2} + \sqrt{14}\right)$

$\Rightarrow \quad 2a + \sqrt{7}a = 4\left(2\sqrt{2} + \sqrt{14}\right)$

$\Rightarrow \quad a\left(2 + \sqrt{7}\right) = 4\sqrt{2}\,(2 + \sqrt{7})$

$\Rightarrow \quad a = 4\sqrt{2}$

$\Rightarrow \quad b = 2\sqrt{14}$

$\therefore \quad a^2 + b^2 = \left(4\sqrt{2}\right)^2 + \left(2\sqrt{14}\right)^2$

$= 32 + 56 = 88$

29. Correct answer is [28].

Explanation: Given: $P_1 : \vec{r}\,.\left(2\hat{i} + \hat{j} - 3\hat{k}\right) = 4$

$\Rightarrow P_1 : 2x + y - 3z = 4$

Now, equation of plane passing through points (2, – 3, 2), (2, – 2, – 3) and (1, – 4, 2) is given by

$$\begin{vmatrix} (x-2) & (y+3) & (z-2) \\ (2-2) & (-2+3) & (-3-2) \\ (1-2) & (-4+3) & (2-2) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y+3 & z-2 \\ 0 & 1 & -5 \\ -1 & -1 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (x - 2)(-5) - (y + 3)(-5) + z - 2) = 0$

$\Rightarrow \quad -5x + 5y + z + 23 = 0$

$\therefore \quad P_2 : -5x + 5y + z + 23 = 0$

Let a, b, c be the direction ratios of the line of intersection of plane P_1 and P_2

$\therefore \quad \frac{a}{1+15} = \frac{-b}{2-15} = \frac{c}{10+5} = \lambda$

$\Rightarrow \quad a = 16\lambda, b = 13\lambda, c = 15\lambda$

$\Rightarrow \quad \alpha = 13, \beta = 15$

$\therefore \quad \alpha + \beta = 28$

Hint :

(i) Equation of plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

(ii) Line of intersection of the planes is perpendicular to the both normal vector of planes.

Shortcut : Given : $P_1 : \vec{r}\,.\left(2\hat{i} + \hat{j} - 3\hat{k}\right) = 4$

$\Rightarrow \quad P_1 : 2x + y - 3z = 4$

$$\text{And} \quad P_2 : \begin{vmatrix} x-2 & y+3 & z-2 \\ 0 & 1 & -5 \\ -1 & -1 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad P_2 : -5x + 5y + z + 23 = 0$

Let a, b, c be direction ratios of the line of intersection of plane P_1 and P_2

$\therefore \quad \frac{a}{1+15} = \frac{-b}{2-15} = \frac{c}{10+c} = \lambda$

$\Rightarrow \quad a = 16\lambda, b = 13\lambda, c = 15\lambda$

$\Rightarrow \quad \alpha = 13, \beta = 15$ Þ $\alpha + \beta = 28$

30. Correct answer is [18915].

Explanation: $b_i \in \{1, 2, 3, ..., 100\}$

Let $\quad P$ = set when b_1, b_2, b_3 are consecutive.

$\therefore \quad n(P) = \underbrace{97 + 97 + 97 + ... 97}_{98 \text{ times}} = 97 \times 98$

Let $\quad \theta$ = set when b_2, b_3, b_4 are consecutive.

$\therefore \quad n(Q) = \underbrace{97 + 97 + ... 97}_{98 \text{ times}} = 97 \times 98$

Now, $P \cap Q$ = set when b_1, b_2, b_3, b_4 are consecutive.

So, $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

$= 97 \times 98 + 97 \times 98 - 97$

$= 97\,(98 + 98 - 1)$

$= 97\,(195)$

$= 18915$

Hint :
(i) There are 98 sets of three consecutive integer and 97 sets of four consecutive integer.
(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Shortcut : There are 98 sets of there consecutive integer and 97 sets of four consecutive integer.

Number of permutation of $b_1\ b_2\ b_3\ b_4$ = (Number of permutation when b_1, b_2, b_3 are consecutive) + (Number of permutations when $b_2\ b_3\ b_4$ are consecutive) – (Number of permutation when $b_1\ b_2\ b_3\ b_4$ are consecutive)

$= 97 \times 98 + 97 \times 98 - 97$

$= 97\,(98 + 98 - 1)$

$= 18915$

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