JEE (Main) MATHS SOLVED PAPER

2022
29th June Shift 2

Time : 1 Hour **Total Marks : 100**

General Instructions :

1. *In Mathematics Section, there are 30 Questions (Q. no. 1 to 30) having Section A and B.*
2. *Section A consists of 20 multiple choice questions & Section B consists of 10 numerical value type questions. In Section B, candidates have to attempt any five questions out of 10.*
3. *There will be only one correct choice in the given four choices in Section A. For each question for Section A, 4 marks will be awarded for correct choice, 1 mark will be deducted for incorrect choice questions and zero mark will be awarded for not attempted question.*
4. *For Section B questions, 4 marks will be awarded for correct answer and zero for unattempted and incorrect answer.*
5. *Any textual, printed or written material, mobile phones, calculator etc. is not allowed for the students appearing for the test.*
6. *All calculations / written work should be done in the rough sheet is provided with Question Paper.*

Mathematics

Section A

Q. 1. Let α be a root of the equation $1 + x^2 + x^4 = 0$. Then the value of $\alpha^{1011} + \alpha^{2022} - \alpha^{3033}$ is equal to :

(A) 1 **(B)** α
(C) $1 + \alpha$ **(D)** $1 + 2\alpha$

Q. 2. Let arg(z) represent the principal argument of the complex number z.

Then, $|z| = 3$ and $\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$ intersect.

(A) Exactly at one point.
(B) Exactly at two points.
(C) Nowhere
(D) At infinitely many points.

Q. 3. Let $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$. If $B = I - {}^5C_1 (\text{adj } A) + {}^5C_2 (\text{adj } A)^5, -\ldots-{}^5C_5 (\text{adj } A)^5$, then the sum of all elements of the matrix B is

(A) –5 **(B)** –6
(C) –7 **(D)** –8

Q. 4. The sum of the infinite series $1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \frac{35}{6^4} + \frac{51}{6^5} + \frac{70}{6^6} + \ldots$ is equal to :

(A) $\frac{425}{216}$ **(B)** $\frac{429}{216}$
(C) $\frac{288}{125}$ **(D)** $\frac{280}{125}$

Q. 5. The value of $\lim_{x\to\infty} \frac{(x^2-1)\sin^2(\pi x)}{x^4 - 2x^3 + 2x - 1}$ is equal to :

(A) $\frac{\pi^2}{6}$ **(B)** $\frac{\pi^2}{3}$
(C) $\frac{\pi^2}{2}$ **(D)** π^2

Q. 6. Let $f : R \to R$ be a function defined by $f(x) = (x-3)^{n_1}(x-5)^{n_2}$, $n_1, n_2 \in N$.

Then, which of the following is *NOT* true?

(A) For $n_1 = 3, n_2 = 4$, there exists $\alpha \in (3, 5)$ where f attains local maxima.
(B) For $n_1 = 4, n_2 = 3$, there exists $\alpha \in (3, 5)$ where f attains local minima,
(C) For $n_1 = 3, n_2 = 5$, there exists $\alpha \in (3, 5)$ where f attains local maxima,
(D) For $n_1 = 4, n_2 = 6$, there exists $\alpha \in (3, 5)$ where f attains local maxima,

Q. 7. Let f be a real valued continuous function on [0,1] and

$f(x) = x + \int_0^1 (x-t) f(t)\, dt$.

Then, which of the following points (x, y) lies on the curve $y = f(x)$?

(A) (2, 4) **(B)** (1, 2)
(C) (4, 17) **(D)** (6, 8)

Q. 8. If $\int_0^1 (\sqrt{2x}-\sqrt{2x-x^2})\,dx = \int_0^1 \left(1-\sqrt{1-y^2}-\frac{y^2}{2}\right) dy + \int_1^2 \left(2-\frac{y^2}{2}\right) dy + I$ then I equal.

(A) $\int_0^1 \left(1+\sqrt{1-y^2}\right)dy$

(B) $\int_0^1 \left(\frac{y^2}{2}-\sqrt{1-y^2}+1\right)dy$

(C) $\int_0^1 \left(1-\sqrt{1-y^2}\right)dy$

(D) $\int_0^1 \left(\frac{y^2}{2}+\sqrt{1-y^2}+1\right)dy$

Q. 9. If $y = y(x)$ is the solution of the differential equation $(1+e^{2x})\frac{dy}{dx} + 2(1+y^2)e^x = 0$ and $y(0) = 0$, then $6\left(y'(0)+\left(y(\log_e\sqrt{3})\right)\right)^2$ is equal to

(A) 2 **(B)** –2

(C) –4 **(D)** –1

Q. 10. Let $P : y^2 = 4ax$, $a > 0$ be a parabola with focus S. Let the tangents to the parabola P make an angle of $\frac{\pi}{4}$ with the line $y = 3x + 5$ touch the parabola P at A and B. Then the value of a for which A, B and S are collinear is.

(A) 8 Only **(B)** 2 Only

(C) $\frac{1}{4}$ Only **(D)** Any a > 0

Q. 11. Let a triangle ABC be inscribed in the circle $x^2 - \sqrt{2}(x+y)+y^2 = 0$ such that $\angle BAC = \frac{\pi}{2}$. If the length of side AB is $\sqrt{2}$, then the area of the ΔABC is equal to :

(A) $(\sqrt{2}+\sqrt{6})/3$ **(B)** $(\sqrt{6}+\sqrt{3})/2$

(C) $(3+\sqrt{3})/4$ **(D)** $(\sqrt{6}+2\sqrt{3})/4$

Q. 12. Let $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z+3}{-1}$ lie on the plane $px - qy + z = 5$, for $p, q \in R$. The shortest distance of the plane from the origin is :

(A) $\sqrt{\frac{3}{109}}$ **(B)** $\sqrt{\frac{5}{142}}$

(C) $\frac{5}{\sqrt{71}}$ **(D)** $\frac{1}{\sqrt{142}}$

Q. 13. The distance of the origin from the centroid of the triangle whose two sides have the equations.

$x - 2y + 1 = 0$ and $2x - y - 1 = 0$ and whose orthocenter is $\left(\frac{7}{3},\frac{7}{3}\right)$ is :

(A) $\sqrt{2}$ **(B)** 2

(C) $2\sqrt{2}$ **(D)** 4

Q. 14. Let Q be the mirror image of the point $P(1, 2, 1)$ with respect to the plane $x + 2y + 2z = 16$. Let T be a plane passing through the point Q and contains the line $\vec{r} = -\hat{k}+\lambda(\hat{i}+\hat{j}+2\hat{k})$, $\lambda \in R$. Then, which of the following points lies on T ?

(A) (2, 1, 0) **(B)** (1, 2, 1)

(C) (1, 2, 2) **(D)** (1, 3, 2)

Q. 15. Let A, B, C be three points whose position vectors respectively are

$\vec{a} = \hat{i}+4\hat{j}+3\hat{k}$

$\vec{b} = 2\hat{i}+\alpha\hat{j}+4\hat{k}, \alpha \in R$

$\vec{c} = 3\hat{i}-2\hat{j}+5\hat{k}$

If α is the smallest positive integer for which $\vec{a}, \vec{b}, \vec{c}$ are noncollinear, then the length of the median, in ΔABC, through A is:

(A) $\frac{\sqrt{82}}{2}$ **(B)** $\frac{\sqrt{62}}{2}$

(C) $\frac{\sqrt{69}}{2}$ **(D)** $\frac{\sqrt{66}}{2}$

Q. 16. The probability that a relation R from $\{x, y\}$ to $\{x, y\}$ is both symmetric and transitive, is equal to

(A) $\frac{5}{16}$ **(B)** $\frac{9}{16}$

(C) $\frac{11}{16}$ **(D)** $\frac{13}{16}$

Q. 17. The number of values of $\alpha \in N$ such that the variance of 3, 7, 12, α, 43-α is a natural number is:

(A) 0 **(B)** 2

(C) 5 **(D)** infinite

Q. 18. From the base of a pole of height 20 meter, the angle of elevation of the top of a tower is 60°. The pole subtends an angle 30° at the top of tower. Then the height of tower is:

(A) $15\sqrt{3}$ (B) $20\sqrt{3}$
(C) $20 + 10\sqrt{3}$ (D) 30

Q. 19. Negation of the Boolean statement $(p \vee q) \Rightarrow ((\sim r) \vee p)$ is equivalent to

(A) $p \wedge (\sim q) \wedge r$ (B) $(\sim p) \wedge (\sim q) \wedge r$
(C) $(\sim p) \wedge q \wedge r$ (D) $p \wedge q \wedge (\sim r)$

Q. 20. Let $n \geq 5$ be an integer. If $9^n - 8n - 1 = 64\alpha$ and $6^n - 5n = 25\beta$, then α-β is

(A) $1 + {}^nC_2(8-5) + {}^nC_3(8^2-5^2) + ... + {}^nC_n(8^{n-1}-5^{n-1})$
(B) $1 + {}^nC_3(8-5) + {}^nC_4(8^2-5^2) + ... + {}^nC_n(8^{n-2}-5^{n-1})$
(C) ${}^nC_3(8-5) + {}^nC_4(8^2-5^2) + ... + {}^nC_n(8^{n-2} - 5^{n-2})$
(D) ${}^nC_4(8-5) + {}^nC_5(8^2 - 5^2) + ... + {}^nC_n(8^{n-3} - 5^{n-3})$

Section B

Q. 21. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c}$ be a vector such that $\vec{a} + \left(\vec{b} \times \vec{c}\right) = \vec{0}$ and $\vec{b} \cdot \vec{c} = 5$.

Then, the value of $3\left(\vec{c} \cdot \vec{a}\right)$ is equal to______.

Q. 22. Let $y = y(x), x > 1$, be the solution of the differential equation

$$(x-1)\frac{dy}{dx} + 2xy = \frac{1}{x-1}, \text{ with } y(2) = \frac{1+e^4}{2e^4}.$$

If $y(3) = \dfrac{e^\alpha + 1}{\beta e^\alpha}$, then the value of $\alpha + \beta$ is equal to____.

Q. 23. Let 3, 6, 9, 12.......upto 78 terms and 5, 9, 13, 17......upto 59 be two series. Then, the sum of the terms common to both the series is equal to____.

Q. 24. The number of solutions of the equation $\sin x = \cos^2 x$ in the interval (0, 10) is____.

Q. 25. For real number $a, b (a > b > 0)$, let

Area $\left\{(x,y) : x^2 + y^2 \leq a^2 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1\right\} = 30\pi$

Area

$$\left\{(x,y) : x^2 + y^2 \geq b^2 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1\right\}$$

$= 18\pi$ Then the value of $(a-b)^2$ is equal to__.

Q. 26. Let f and g be twice differentiable even functions on (-2, 2) such that

$f\left(\frac{1}{4}\right) = 0, f\left(\frac{1}{2}\right) = 0, f(1) = 1$, and $g\left(\frac{3}{4}\right) = 0, g(1) = 2$

Then, the minimum number of solutions of $f(x)g''(x) + f'(x)g'(x) = 0$ in $(-2, 2)$ is equal to______.

Q. 27. Let the coefficients of x^{-1} and x^{-3} in the expansion of $\left(2x^{\frac{1}{5}} - \frac{1}{x^{\frac{1}{5}}}\right)^{15}$, $x > 0$, be m and n respectively. If r is a positive integer such that $mn^2 = {}^{15}C_r \cdot 2^r$, then the value of r is equal to________ .

Q. 28. The total number of four digit numbers such that each of first three digits is divisible by the last digit, is equal to__________ .

Q. 29. Let $M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$, where α is a non-zero real number an $N = \sum_{k=1}^{49} M^{2k}$. If $(I - M^2)N = -2I$, than the positive integral value of α is____.

Q. 30. Let $f(x)$ and $g(x)$ be two real polynomials of degree 2 and 1 respectively. If $f(g(x)) = 8x^2 - 2x$, and $g(f(x)) = 4x^2 + 6x + 1$, then the value of $f(2) + g(2)$ is____.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
Section (A)			
1	A	Roots of Unity	Complex Numbers
2	C	Modulus and Argument of Complex Number	Complex Numbers
3	C	Adjoint of a Matrix	Matrices and Determinants
4	C	Geometric Progressions	Sequences and Series
5	D	Properties of Limits	Limits
6	C	Maxima and Minima	Application of Derivatives
7	D	Basics of Definite Integrals	Definite Integration
8	C	Basics of Definite Integrals	Definite Integration
9	C	Solution of Differential Equations	Differential Equations
10	D	Tangent to Parabola	Parabola
11	Bonus	Interaction between Circle and a Line	Circle
12	B	Plane and a line	Three Dimensional Geometry
13	C	Special Points in Triangles	Point and Straight Line
14	B	Line and Plane	Three Dimensional Geometry
15	A	Scalar and Vector Products	Vector Algebra
16	A	Basics of Probability	Probability
17	A	Measures of Dispersion	Statistics
18	D	Heights and Distances	Heights and Distances
19	C	Tautology	Mathematical Reasoning
20	C	Multinomial	Binomial
Section (B)			
21	Bonus	Basics of Vectors	Vector Algebra
22	14	Linear Differential Equations	Differential Equations
23	2223	Arithmetic Progressions	Sequences and Series
24	4	Trigonometric Equations	Trigonometric Equations and Inequalities
25	12	Area Bounded by Curves	Area under Curves
26	4	Rules of Differentiation	Differential Coefficient
27	5	Properties of Binomial Coefficients	Binomial Theorem
28	1086	Division and Distribution of Objects	Permutation and Combination
29	1	Algebra of Matrices	Matrices and Determinants
30	18	Composite Functions	Functions

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JEE (Main) MATHEMATICS SOLVED PAPER

2022

29^{th} June Shift 2

ANSWERS WITH EXPLANATIONS

Mathematics

Section A

1. Option (A) is correct.

Explanation: Given : $1 + x^2 + x^4 = 0$

$\Rightarrow x^4 - x^2 + x^2 + x^2 + x - x + 1 = 0$

$\Rightarrow (x^2 - x)(x^2 + x) + (x^2 - x) + (x^2 + x + 1) = 0$

$\Rightarrow (x^2 - x)(x^2 + x + 1) + (x^2 + x + 1) = 0$

$\Rightarrow (x^2 + x + 1)(x^2 - x + 1) = 0$

$\Rightarrow x^2 + x + 1 = 0$ or $x^2 - x^2 + 1 = 0$

$\Rightarrow x = \omega, \omega^2$ or $x = \omega, \omega^2$, where ω is a cube root of unity.

$\Rightarrow \alpha = \omega$

Now, $\alpha^{1011} + \alpha^{2022} - \alpha^{3033} = \omega^{1011} + \omega^{2022} - \omega^{3033}$

$= (\omega^3)^{337} + (\omega^3)^{674} - (\omega^3)^{1011}$

$= 1 + 1 - 1 \quad \{\because \omega^3 = 1\}$

$= 1$

Hint:

(i) use $1 + x^2 + x^4 = (x^2 + x + 1)(x^2 - x + 1)$

(ii) If ω is a cube root of unity, then $\omega^3 = 1$

Shortcut :

Given: $1 + x^2 + x^4 = 0$

$\Rightarrow (x^2 + x + 1)(x^2 - x + 1) = 0$

$\Rightarrow x = \pm\omega, \pm\omega^2$, where ω is cube root of unity.

$\therefore \alpha = \omega$

Now, $\alpha^{1011} + \alpha^{2022} - \alpha^{3033} = 1 + 1 - 1 = 1$

2. Option (C) is correct.

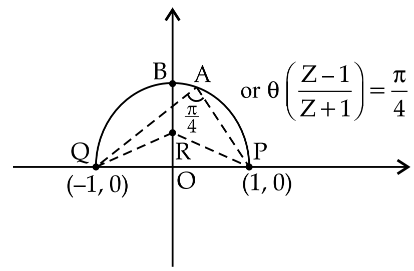
Explanation: Given: $|z| = 3$

$\Rightarrow$ It represents a circle of radius 3 and centre at (0, 0)

And $\arg(z - 1) - \arg(z + 1) = \dfrac{\pi}{4}$

$\Rightarrow$ agr $\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$

$\Rightarrow z$ is on major are of circle having PQ as chord and $R(0, a)$ as centre of the circle.



So, $\angle PAQ = \angle ORP = \dfrac{\pi}{4}$

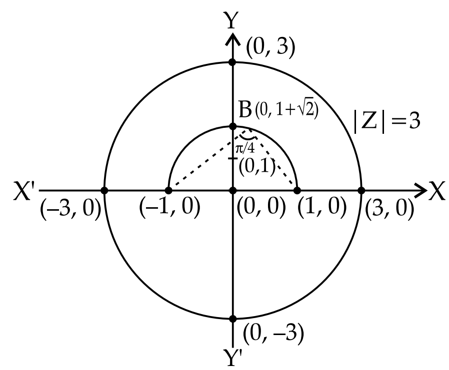
$\Rightarrow OP = OR = a = 1$

$\therefore$ Coordinates of R is (0,1)

So, radius $= RP = \sqrt{2}$

$\therefore OB = OR + \sqrt{2} = 1 + \sqrt{2}$

$\Rightarrow B = (0, 1 + \sqrt{2})$



So, it is clear from figure, both curves do not intersect.

$\therefore$ No z satisfy both the equation.

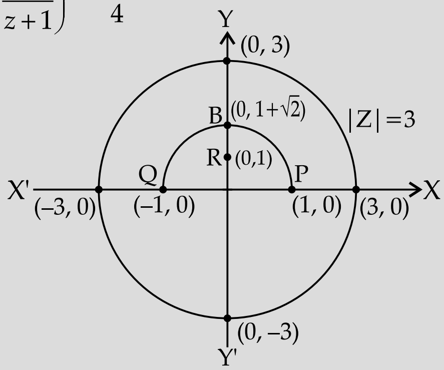
Hint:

(i) $|z| = r$ represents a circle of radius r and centre at (0, 0)

(ii) arg $= \left(\dfrac{z-a}{z+a}\right) = Q$; $\theta \in \left(0, \dfrac{\pi}{2}\right)$, $a > 0$ represents major are of circle having PQ as a chord and $R(0, b)$ as centre of the circle, where $P(a, o)$ and $Q(-a, o)$

Shortcut:

Lets draw the curve for $|z| = 3$ and arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$



$\therefore$ No Z satisfy both the equation.

3. Option (C) is correct.

Explanation: Given $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

And $B = I - 5_{C_1}(\text{adj } A) + 5_{C_2}(\text{adj } A)^2 \ldots 5_{C_5}(\text{adj } A)^5$

$\Rightarrow \quad B = (I - \text{adj } A)^5$

Now, $\quad \text{adj } A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\therefore \quad B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^5$

$\Rightarrow \quad B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^5$

$\Rightarrow \quad B = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^5$

New $\quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Sim: Jarly, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

So, $\quad B = -\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}$

$\therefore$ Modulus of sum of all elements of matrix $B = |-1-5-1| = 7$

Hint:

(i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{adj }(A) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$

(ii) Use $(a-b)^n = {}^nC_0a^{n} - {}^nC_1C^{n-1}\,b + {}^nC_2\,a^{n-2}\,b^2 + \ldots.$

4. Option (C) is correct.

Explanation:

Let $\quad P = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \frac{35}{6^4} + \ldots. \quad$ ...(i)

$\Rightarrow \quad \frac{P}{6} = \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \frac{22}{6^4} + \ldots. \quad$ ...(i)

Equation (i) – Equation (ii), we get

$$P - \frac{P}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \frac{13}{6^4} + \ldots$$

$\Rightarrow \quad \frac{5P}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \frac{13}{6^4} + \ldots \quad$ ...(iii)

$\Rightarrow \quad \frac{5P}{6^2} = \frac{1}{6} + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + \ldots. \quad$ ...(iv)

Equation (iii) – Equation (iv), we get

$$\frac{5P}{6} - \frac{5P}{6^2} = 1 + \frac{3}{6} + \frac{3}{6^2} + \frac{3}{6^3} + \frac{3}{6^4} + \ldots$$

$\Rightarrow \quad \left(\frac{5}{6}\right)^2 P = 1 + 3\left(\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \ldots\right)$

$\Rightarrow \quad \left(\frac{5}{6}\right)^2 P = 1 + 3\left(\frac{\frac{1}{6}}{1 - \frac{1}{6}}\right)$

$\Rightarrow \quad \left(\frac{5}{6}\right)^2 P = 1 + \frac{3}{5} = \frac{8}{5}$

$\Rightarrow \quad P = \frac{8}{5} \times \frac{36}{25} = \frac{288}{125}$

Hint:

(i) Convert the given series in form of infinite *G.P.* and solve further.

(ii) Sum of infinite *GP* whose first term is a and common ratio is r, given by $a/1-r$; $r<1$

Shortcut:

Let $\quad P = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \frac{35}{6^4} \ldots.$

$\Rightarrow \quad \frac{P}{6} = \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \frac{22}{6^4} + \ldots.$

$\Rightarrow \quad \frac{5P}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \ldots$

$\Rightarrow \quad \frac{5P}{6^2} = \frac{1}{6} + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + \ldots$

$$\Rightarrow \quad \left(\frac{5}{6}\right)^2 P = 1 + 3\left(\frac{1}{6}+\frac{1}{6^2}+\frac{1}{6^3}+....\right)$$

$$\Rightarrow \quad \left(\frac{5}{6}\right)^2 P = 1+3\left(\frac{\frac{1}{6}}{1-\frac{1}{6}}\right)$$

$$\Rightarrow \quad P = \frac{288}{125}$$

5. Option (D) is correct.

Explanation:

Let $A = \lim\limits_{x\to1}\dfrac{(x^2-1)\sin^2(\pi x)}{x^4-2x^3+2x-1}\left(\dfrac{0}{0}\text{form}\right)$

$$\Rightarrow \quad A = \lim_{x\to1}\frac{(x^2-1)\sin^2\pi x}{(x^4-1)-(2x^3-2x)}$$

$$\Rightarrow \quad A = \lim_{x\to1}\frac{(x^2-1)\sin^2\pi x}{(x^2-1)(x^2+1)-2x(x^2-1)}$$

$$\Rightarrow \quad A = \lim_{x\to1}\frac{(x^2-1)\sin^2\pi x}{(x^2-1)(x^2+1-2x)}$$

$$\Rightarrow \quad A = \lim_{x\to1}\frac{\sin^2\pi x}{(x-1)^2}$$

Put $x = 1 + h$

$$\Rightarrow \quad A = \lim_{h\to0}\frac{\sin^2\pi(1+h)}{h^2} = \lim_{h\to0}\frac{(-\sin\pi h)^2}{h^2}$$

$$\Rightarrow \quad A = \lim_{x\to1}\left(\frac{\sin\pi h}{\pi h}\right)^2.\pi^2$$

$$\Rightarrow \quad A = \pi^2\left\{\because \lim_{x\to0}\frac{\sin x}{x}=1\right\}$$

Hint:

(i) Use $x^4-2x^3+2x-1 = (x^2-1)(x-1)^2$

(ii) Use $\lim\limits_{x\to0}\dfrac{\sin x}{x}=1$

Shortcut:

Let $A = \lim\limits_{x\to1}\dfrac{(x^2-1)\sin^2\pi x}{x^4-2x^3+2x-1}$

$$\Rightarrow \quad A = \lim_{x\to1}\frac{(x^2-1)\sin^2\pi x}{(x^2-1)(x+1)^2}$$

$$\Rightarrow \quad A = \lim_{x\to1}\left(\frac{\sin\pi(1-x)}{\pi(1-x)}\right)^2.\pi^2$$

$$\Rightarrow \quad A = \pi^2$$

6. Option (C) is correct.

Explanation: Given : $f(x) = (x-3)^{n_1}(x-5)^{n_2}$

$\Rightarrow f'(x) = (x-3)^{n_1}\{n_2(x-5)^{n_2-1}\} + (x-5)^{n_2}\{n_1(x-3)^{n_1-1}\}$

$\Rightarrow f'(x) = (x-3)^{n_1-1}(x-5)^{n_2-1}\{n_2(x-3)+n_1(x-5)\}$

$\Rightarrow f'(x) = (x-3)^{n_1-1}(x-5)^{n_2-1}\{(n_1+n_2)x-5n_1+3n_2)\}$

$\Rightarrow f'(x) = (x-3)^{n_1-1}(x-5)^{n_2-1}(n_1+n_2)\left(x-\dfrac{5n_1+3n_2}{n_1+n_2}\right)$

For $n_1 = 3, n_2 = 4$

$\Rightarrow f'(x) = 7(x-3)^2(x-5)^3\left(x-\dfrac{27}{7}\right)$

+   +   −   +

$x=3$ $x=\dfrac{27}{7}$ $x=5$

Since, sign of $f'(x)$ changes from positive to negative at $x = \dfrac{27}{7}$

$\therefore f'(x)$ has local maxima at $x = \dfrac{27}{7} \in (3, 5)$

For $n_1 = 4, n_2 = 3$

$f'(x) = 7(x-3)^3(x-5)^2\left(x-\dfrac{29}{7}\right)$

+   −   +   +

$x=3$ $x=\dfrac{29}{7}$ $x=5$

Since, sign of $f'(x)$ changes from negative to positive at $x = \dfrac{29}{7}$

$\therefore f(x)$ has local minima at x = $\dfrac{29}{7} \in (3,5)$

For $n_1 = 3, n_2 = 5$

$f'(x) = 8(x-3)^2(x-5)^4\left(x-\dfrac{30}{8}\right)$

−   −   +   +

$x=3$ $x=\dfrac{29}{7}$ $x=5$

Since, sign of $f'(x)$ changes from negative to positive at $x = \dfrac{30}{8}$

$\therefore f(x)$ has local minima at x = $\dfrac{30}{8} \in (3,5)$

For $n_1 = 4, n_2 = 6$

$f'(x) = 10(x-3)^2(x-5)^5\left(x-\dfrac{38}{8}\right)$

– + – +

$x=3$ $x = \frac{38}{10}$ $x=5$

Since, sign of $f'(x)$ changes from negative to positive at $x = \frac{38}{8}$

$\therefore f(x)$ has local minima at $x = \frac{38}{8} \in (3,5)$

Hint:

(i) Find $f'(x)$ using product rule of differentiation and expression for $f'(x)$ for different values of n_1 and n_2 as per options and solve further using first derivative test.

(ii) $(UV)' = u'v + v'u$

Shortcut:

Given : $f(x) = (x-3)^{n_1}(x-5)^{n_2}$

$\Rightarrow f'(x) = (x-3)^{n_1-1}(x-5)^{n_2-1}(n_1+n_2)\left(x - \frac{5n_1+3n_2}{n_1+n_2}\right)$

For $n_1 = 3, n_2 = 5$

$f'(x) = 8(x-3)2(x-5)^4\left(x - \frac{30}{8}\right)$

– – + +

$x=3$ $x = \frac{30}{8}$ $x=5$

$\therefore f(x)$ has local minima at $x = \frac{30}{8} \in (3,5)$

So, option (3) is not true.

7. Option (D) is correct.

Explanation: Given : $f(x) = x + \int_0^1 (x-t)f(t)\,dt$

$\Rightarrow f(x) = \left(1 + \int_0^1 f(t)\,dt\right)x - \int_0^1 t\,f(t)\,dt$

$\Rightarrow f(x) = Px - Q$, where

$P = 1 + \int_0^1 f(t)\,dt$ and $Q = \int_0^1 tf(t)\,dt$

Now, $P = 1 + \int_0^1 (Pt-Q)\,dt$

$\Rightarrow P = 1 + \left[\frac{Pt^2}{2} - Qt\right]_0^1$

$\Rightarrow P = 1 + \frac{P}{2} - Q$

$\Rightarrow P = 2(1-Q)$ ... (i)

$\therefore Q = \int_0^1 tf(t)\,dt$

$\Rightarrow Q = \int_0^1 t(Pt-Q)\,dt$

$\Rightarrow Q = \left[\frac{Pt^3}{3} - \frac{Qt^2}{2}\right]_0^1$

$\Rightarrow Q = \frac{P}{3} - \frac{Q}{2}$

$\Rightarrow Q = \frac{2P}{9}$ ...(ii)

On Solving equation (i) and equation (ii), we get

$P = \frac{18}{13}, Q = \frac{4}{13}$

$\therefore f(x) = \frac{18}{13}x - \frac{4}{13}$

Since, (6,8) satisfy the equation of $f(x) = \frac{18}{13}x - \frac{4}{13}$

$\therefore$ Point (6,8) lies on the carve $y = f(x)$

Hint:

(i) Covert given functional equation in form of $f(x) = Px - Q$ and solve further.

(ii) Use. $\int_a^b f(x)dx = [F(x)]_a^b$,

where $\int f(x)dx = F(x) + c$

Shortcut:

$f(x) = \left(1 + \int_0^1 f(t)\,dt\right)x - \int_0^1 tf(t)\,dt$

$\Rightarrow$ $f(x) = Px - Q$, where

$P = 1 + \int_0^1 f(t)\,dt$ and $Q = \int_0^1 tf(t)\,dt$

$\Rightarrow$ $P = 2(1-Q)$ and $Q = \frac{2P}{9}$

$\Rightarrow$ $P = \frac{18}{13}$ and $Q = \frac{4}{13}$

$\therefore$ $f(x) = \frac{18}{13}x - \frac{4}{13}$

$\because$ $f(6) = 8$

$\therefore$ Point (6, 8) lies on the curve $y = f(x)$.

8. Option (C) is correct.

Explanation:

$$\int_0^2 (\sqrt{2x} - \sqrt{2x - x^2})\, dx = \int_0^1 (1 - \sqrt{1-y^2} - \frac{y^2}{2})\, dy + \int_1^2 (2 - \frac{y^2}{2})\, dy + \text{I}$$

Now, $L.H.S. = \int_0^2 \sqrt{2x}\, dx - \int_0^2 \sqrt{2x - x^2}\, dx$

$$= \sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2 - \int_0^2 \sqrt{1-(x-1)^2}\, dx$$

$$= \sqrt{2}\left[\frac{4\sqrt{2}}{3} - 0\right] - 2\int_0^1 \sqrt{1-y^2}\, dy$$

$$= \frac{8}{3} - 2\int_0^1 \sqrt{1-y^2}\, dy$$

And $R.H.S. = \int_0^1 (1 - \frac{y^2}{2})\, dy - \int_0^1 \sqrt{1-y^2}\, dy + \int_1^2 (2 - \frac{y^2}{2})\, dy + \text{I}$

$$= \left[y - \frac{y^3}{6}\right]_0^1 - \int_0^1 \sqrt{1-y^2}\, dy + \left[2y - \frac{y^3}{6}\right]_1^2 + \text{I}$$

$$= \left[1 - \frac{1}{6}\right] - \int_0^1 \sqrt{1-y^2}\, dy + \left[4 - \frac{8}{6} - 2 + \frac{1}{6}\right] + \text{I}$$

$$= \frac{5}{6} - \int_0^1 \sqrt{1-y^2}\, dy + \frac{5}{6} + \text{I}$$

$$= \frac{5}{3} - \int_0^1 \sqrt{1-y^2}\, dy + \text{I}$$

So, $\frac{8}{3} - 2\int_0^1 \sqrt{1-y^2}\, dy = \frac{5}{3} - \int_0^1 \sqrt{1-y^2}\, dy + \text{I}$

$\Rightarrow \quad \text{I} = 1 - \int_0^1 \sqrt{1-y^2}\, dy$

$\Rightarrow \quad \text{I} = \int_0^1 (1 - \sqrt{1-y^2})\, dy$

Hint:

(i) Simplify given expression using method of substitution and solve further.

(ii) Use $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx \,;\, f(-x) = f(x)$

(iii) $\int_a^b f(x)\, dx = F(b) - F(a)$, where $\int f(x)\, dx = F(x)$

Shortcut:

Given: $\int_0^2 (\sqrt{2x} - \sqrt{2x - x^2})\, dx$

$$= \int_0^1 (1 - \frac{y^2}{2} - \sqrt{1-y^2})\, dy + \int_1^2 (2 - \frac{y^2}{2})\, dy + \text{I}$$

$$L.H.S. = \int_0^2 \sqrt{2x}\, dx - \int_0^2 \sqrt{1-(x-1)^2}\, dx$$

$$= \frac{8}{3} - 2\int_0^1 \sqrt{1-y^2}\, dy$$

And $R.H.S. = \int_0^1 (1 - \frac{y^2}{2})\, dy - \int_0^1 \sqrt{1-y^2}\, dy + \int_1^2 (2 - \frac{y^2}{2})\, dy + \text{I}$

$$= \frac{5}{6} - \int_0^1 \sqrt{1-y^2}\, dy + \frac{5}{6} + \text{I}$$

$$= \frac{5}{3} - \int_0^1 \sqrt{1-y^2}\, dy + \text{I}$$

So, $\frac{8}{3} - 2\int_0^1 \sqrt{1-y^2}\, dy = \frac{5}{3} - \int_0^1 \sqrt{1-y^2}\, dy + \text{I}$

$\Rightarrow \quad \text{I} = 1 - \int_0^1 \sqrt{1-y^2}\, dy$

$\Rightarrow \quad \text{I} = \int_0^1 (1 - \sqrt{1-y^2})\, dy$

9. Option (C) is correct.

Explanation: $(1 + e^{2x})\frac{dy}{dx} + 2(1 + y^2)\, e^x = 0$ and $y(0) = 0$

$\Rightarrow \quad \frac{dy}{1+y^2} = -\frac{2e^x}{1+e^{2x}}\, dx$

$\Rightarrow \quad \int \frac{dy}{1+y^2} = -2\int \frac{e^x}{1+e^{2x}}\, dx$

$\Rightarrow \quad \tan^{-1}(y) = -2\tan^{-1} e^x + C$

$\because \quad y(0) = 0$

$\therefore \quad \tan^{-1}(0) = -2\tan^{-1}(1) + C$

$\Rightarrow \quad C = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1} y + 2\tan^{-1} e^x = \frac{\pi}{2} \quad \ldots\text{(i)}$

Now, $\left(\frac{dy}{dx}\right)_{x=0} = \left(-\frac{2(1+y^2)e^x}{1+e^{2x}}\right)_{x=0}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=0} = -1$$

Put $x = \log_e \sqrt{3}$ in equation (*i*), we get

$$\tan^{-1} y + 2\tan^{-1} e^{\log_e \sqrt{3}} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y + 2\tan^{-1} \sqrt{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y + \frac{2\pi}{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y = -\frac{\pi}{6}$$

$$\Rightarrow \quad y(\log_e \sqrt{3}) = -\frac{1}{\sqrt{3}}$$

$$\therefore 6[y'(0) + y(\log_e \sqrt{3})^2] = 6\left[-1+\frac{1}{3}\right] = -4$$

Hint:

(i) Apply variable separable method for solving given differential education.

(ii) Use $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Shortcut:

Given:

$$(1 + e^{2x})\frac{dy}{dx} + 2(1 + y^2)e^x = 0$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = \int -\frac{2e^x}{1+e^{2x}}\,dx$$

$$\Rightarrow \quad \tan^{-1} y = -2\tan^{-1} e^x + C$$

$$\because \quad y(0) = 0$$

$$\Rightarrow \quad C = \frac{\pi}{2}$$

$$\therefore \quad \tan^{-1} y = -2\tan^{-1} e^x + \frac{\pi}{2}$$

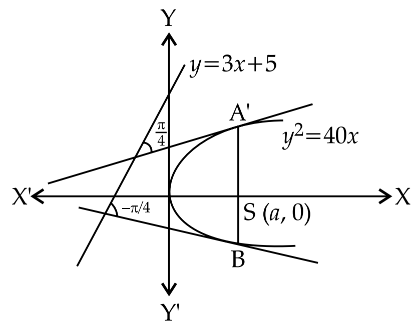
Now, $\left(\frac{dy}{dx}\right)_{x=0} = -1$ and $y(\log_e \sqrt{3}) = -\frac{1}{\sqrt{3}}$

$$\therefore 6[y'(0) + y(\log_e \sqrt{3})^2] = 6\left[-1+\frac{1}{3}\right] = -4$$

10. Option (D) is correct.

Explanation: $p : y^2 = 40x, a > 0$

And tangents to the parabola makes an angle of $\frac{\pi}{4}$ with $y = 3x + 5$



Let slope of tangent be m.

$$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{m-3}{1+3m}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{m-3}{1+3m}\right|$$

$$\Rightarrow \quad \left|\frac{m-3}{1+3m}\right| = \pm 1$$

$$\Rightarrow \quad \frac{m-3}{1+3m} = 1 \text{ and } \frac{m-3}{1+3m} = -1$$

$$\Rightarrow \quad m = -2 \text{ and } m = \frac{1}{2}$$

$\therefore$ Point of contact are $B\left(\frac{a}{(-2)^2}, \frac{2a}{(-2)}\right)$ and $A\left(\frac{a}{\left(\frac{1}{2}\right)^2}, \frac{2a}{\frac{1}{2}}\right)$

$\Rightarrow B\left(\frac{a}{4}, -a\right)$ and $A(4a, 4a)$

$\because$ Points A, S and B are collinear.

$$\Rightarrow \quad \begin{vmatrix} 4a & 4a & 1 \\ \frac{a}{4} & -a & 1 \\ a & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4a(-a) - 4a\left(\frac{a}{4} - a\right) + 1(a^2) = 0$$

$$\Rightarrow \quad -4a^2 + 3a^2 + a^2 = 0$$

$$\Rightarrow \quad 0 = 0$$

$\therefore$ Points A, S and B are always collinear for $A \in R$.

Hint:

(i) The angle between two lines with slopes m_1 and m_2 is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

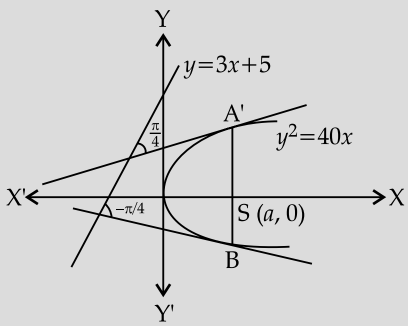
(ii) The equation of tangent to the parallel $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ and the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Shortcut:

Given: $p : y^2 = 40x, a > 0$

And line: $y = 3x + 5$



Let slope of tangent be m.

$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{m-3}{1+3m}\right|$

$\Rightarrow \quad m = -2, \frac{1}{2}$

$\therefore$ Point of contacts are $B\left(\frac{a}{4}, -a\right)$ and $A\,(4a, 4a)$

$\because$ Ponts A, S and B are collinear.

$$\therefore \quad \begin{vmatrix} 4a & 4a & 1 \\ a & 0 & 1 \\ \frac{a}{4} & -a & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 0 = 0$

$\therefore$ Points A, S and B are always collinear for $a \in R$.

11. Correct answer is (1) {None of the option is correct}

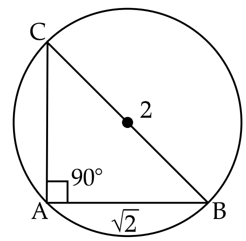
Note: Students should be awarded the marks who has attempted this question.

Explanation:

Equation of circle is $x^2 - \sqrt{2}\,(x + y) + y^2 = 0$

$\therefore$ Coordinates of centre of circle is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

and radius is $\sqrt{\frac{1}{2} + \frac{1}{2} - 0} = 1$



$\therefore \quad BC = ?$ (length of diameter of circle)

$\Rightarrow \quad BC = 2$

Apply pythogores theorem in ΔABC, we get

$AC^2 + AB^2 = BC^2$

$\Rightarrow \quad AC^2 = 4 - 2 = 2$

$\Rightarrow \quad AC = \sqrt{2}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times AB \times AC$

$= \frac{1}{2} \times \sqrt{2} \times \sqrt{2}$

$= 1$ square unit.

Hint:

(i) Hypotenuse of ΔACB will be passes through the centre of given circle.

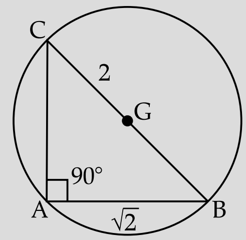
(ii) Coordinates of centre of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Shortcut:

(i) Hypotenuse of ΔACB will be passes through the centre of given circle.

Given: $x^2 - \sqrt{2}\,(x + y) + y^2 = 0$

$\Rightarrow$ Centre $= G\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and radius $= 1$



Now, BC = lenght of diameter of circle = 2

And $\quad AC = \sqrt{BC^2 - AB^2} = \sqrt{2}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\,(\sqrt{2})\,(\sqrt{2}) = 1$ square unit.

12. Option (B) is correct.

Explanation: Given: Line $L : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z+3}{-1}$

And plane $P : px - qy + z = 5$

$\because$ Line L lies on the plane p

$\therefore$ Point (2, – 1, – 3) will satisfy th equation of plane

So, $2p + q - 3 = 5$

$\Rightarrow \quad 2p + q = 8 \quad$ ...(i)

The line is also parallel to the plane.

$\therefore \quad 3p - 2q - 1 = 0$

$\Rightarrow \quad 3p - 2q = 1 \quad$ ...(ii)

On solving equation (i) and equation (ii), we get

$$p = 15, q = -22$$

$\therefore$ Equation of plane is $15x + 22y + z - 5 = 0$

Now, distance of plane from origin is

$$d = \left|\frac{-5}{\sqrt{(15)^2 + (22)^2 + 1^2}}\right|$$

$$\Rightarrow \quad d = \frac{5}{\sqrt{710}} = \sqrt{\frac{25}{710}} = \sqrt{\frac{5}{142}}$$

Hint:

(i) If line lies on the plane, then line is parallel to the plane and plane contains all the points which lies on the line.

(ii) Distance of a plane $ax + by + cz + d = 0$ from origin is $\left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

Shortcut:

Given : Line L : $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z+3}{-1}$

And plane $P : px - qy + z = 5$

Line L lies on the plane p

$\therefore \quad 2p + q - 3 = 5$

and $\quad 3p + 2q - 1 = 0$

$\Rightarrow \quad p = 15, q = -22$

$\therefore$ Equation of plane is $15x + 22y + z = 5$

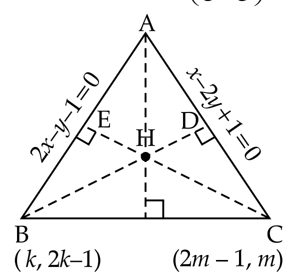
Now, distance of plane from origin

$$= \left|\frac{-5}{\sqrt{(15)^2 + (22)^2 + 1}}\right|$$

$$= \sqrt{\frac{5}{142}}$$

13. Option (C) is correct.

Explanation: Given : Coordinates of ortho centre of triangle is $H = \left(\frac{7}{3}, \frac{7}{3}\right)$



Let equation of line $AB : 2x - y - 1 = 0 \quad$ ...(i)

And equation of line $AC : x - 2y + 1 = 0 \quad$ ...(ii)

On solving equation (i) and equation (ii), we get $x = 1, y = 1$

$\therefore$ Coordinates of point A is (1, 1)

Let coordinates of point B be $(k, 2k - 1)$ and C be $(2m - 1, m)$

Now, slope of $AC \times$ slope of $BD = -1$

$$\Rightarrow \left(\frac{1}{2}\right)\left(\frac{\frac{7}{3} - 2k + 1}{\frac{7}{3} - k}\right) = -1$$

$\Rightarrow \quad k = 2$

Now, slope of $AB \times$ slope of $CE = -1$

$$\Rightarrow \quad (2) = \left(\frac{\frac{7}{3} - m}{\frac{7}{3} - 2m + 1}\right) = -1$$

$\Rightarrow \quad m = 2$

$\therefore$ A (1, 1), B (2, 3), C (3, 2)

Now, coordinates of centroid

$$G = \left(\frac{1+2+3}{3}, \frac{1+3+2}{3}\right) = (2, 2)$$

$$\therefore \quad OG = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

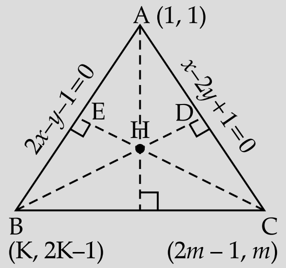
Hint:

(i) The orthocentre of a triangle is the point where the perpendicular drawn from the vertices to the opposite sides of the triangle intersect each other.

(ii) For perpendicular lines, product of their slopes is equal to – 1.

Shortcut:

Given : $H\left(\frac{7}{3}, \frac{7}{3}\right)$



$\because BD \perp AC$

$\therefore \quad (m)_{BD}\,(m)_{AC} = -1$

$\Rightarrow \quad \left(\frac{1}{2}\right)\left(\frac{\frac{7}{3} - 2k + 1}{\frac{7}{3} - k}\right) = -1$

$\Rightarrow \quad k = 2$

Also, $(m)_{CE}\,(m)_{AB} = -1$

$\Rightarrow \quad m = 2$

$\therefore$ $A(1, 1)$, $B(2, 3)$, $C(3, 2)$

So, coordinates of centroid is

$$G = \left(\frac{1+2+3}{3}, \frac{1+3+2}{3}\right) = (2, 2)$$

$\therefore \quad OG = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

14. Option (B) is correct.

Explanation: Given: Plane: $x + 2y + 2z - 16 = 0$

And line: $\vec{r} = -\hat{k} + \lambda(\hat{i} + \hat{j} + 2\hat{k})$

So, equation of line in symmetric form is

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z+1}{2}$$

Now, mirror image of $p(1, 2, 1)$ in plane $x + 2y + 2z - 16 = 0$ is

$$\Rightarrow \quad \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{2}$$

$$= -2\left(\frac{1 + 2(2) + 2(1) - 16}{1^2 + 2^2 + 2^2}\right)$$

$$\Rightarrow \quad \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{2} = 2$$

$\Rightarrow \quad x = 3, y = 6, z = 5$

$\therefore$ Coordinates of point Q are $(3, 6, 5)$

Now, equation of plane T is $\begin{vmatrix} x & y & z+1 \\ 1 & 1 & 2 \\ 3 & 6 & 6 \end{vmatrix} = 0$

$\Rightarrow \quad x(6-12) - y(6-6) + (z+1)(6-3) = 0$

$\Rightarrow \quad -6x + 3z + 3 = 0$

$\Rightarrow \quad 2x - z - 1 = 0$

So, by options (1, 2, 1) lies on plane T.

Hint:

(i) The coordinates of the image of point (x_1, y_1, z_1) w.r.t. plane $ax + by + cz + d = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -2\,\frac{(ax_1 + by_1 + cz_1 + b)}{a^2 + b^2 + c^2}$$

(ii) Equation of plane containing the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ and passing through the point (x_2, y_2, z_2) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a & b & c \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \end{vmatrix} = 0$$

Shortcut:

Image of $p(1, 2, 1)$ in $x + 2y + 2z - 16$ is $Q(3, 6, 5)$

equation of plane T is given by

$$\begin{vmatrix} x & y & z+1 \\ 1 & 1 & 2 \\ 3 & 6 & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad 2x - z = 1$

By options, (1, 2, 1) lies on plane T

15. Option (A) is correct.

Explanation: Given: position vectors of three point A, B, C are

$$\vec{a} = \hat{i} + 4\hat{j} + 3\hat{k}$$

$$\vec{b} = 2\hat{i} + \alpha\,\hat{j} + 4\hat{k}$$

$$\vec{c} = 3\hat{i} - 2\hat{j} + 5\hat{k}$$

Now, $\quad \vec{AB} = \vec{b} - \vec{a}$

$$= \hat{i} + (\alpha - 4)\hat{j} + \hat{k}$$

And $\quad \overrightarrow{AC} = \vec{c} - \vec{a}$

$$= 2\hat{i} - 6\hat{j} + 2\hat{k}$$

If A, B, C are collinear, then $\overrightarrow{AB} \parallel \overrightarrow{AC}$

$$\Rightarrow \quad \frac{1}{2} = \frac{\alpha - 4}{-6} = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = 1$$

$\therefore$ $a = 2$ is the smallest positive integer for which A, B, C are non-collinear.

Now, mid-point of $BC = p\left(\frac{5}{2}, 0, \frac{9}{2}\right)$

$\therefore$ Length of median through $A = AP$

$$= \sqrt{\left(\frac{5}{2} - 1\right)^2 + (4)^2 + \left(\frac{9}{2} - 3\right)^2}$$

$$= \sqrt{\frac{9}{4} + 16 + \frac{9}{4}}$$

$$= \frac{\sqrt{82}}{2}$$

Hint:

Find the value of α for which points are collinear using $\overrightarrow{AB} \parallel \overrightarrow{AC}$ and choose smallest positive integer value of α for which given points are non-collinear.

16. Option (A) is correct.

Explanation: Let set $p = \{x, y\}$

$\because \quad = \{x, y\}$

$\Rightarrow p \times Q = \{(x, x), (x, y\}, (y, x), (y, y)\}$

So, total number of relation from p to $Q = 2^4 = 16$

Now, relation which are symmetric as well is transitive are ϕ, $\{x, x\}$, $\{(y, y\}$, $\{(x, x), (y, y)\}$, $\{(x, x), (x, y), (y, y), (y, x)\}$

$\therefore$ Fovorable case = 5

As we know by classic definition of probability, the probability of on event is the ratio of the number of cases favourcable to it, to the number of total possible cases.

$\therefore$ required probability $= \frac{5}{16}$

Hint:

(i) If $n(A) = p$, then number of relation from set A to A is 2^{p^2}.

(ii) Recall the definition of symmetric and transitive relation.

(iii) The probability of on event is the ratio of the number of cases favourable to it, to the number of total possible cases.

Shortcut:

Total number of relation $= 2^{2^2} = 16$

Favourable relation $= \phi$, $\{(x, x)\}$, $\{(y, y)\}$, $\{(x, x)\}$, $\{(x, x), (x, y), (y, y), (y, x)\}$

$\therefore$ Required probability $= \frac{5}{16}$.

17. Option (A) is correct.

Explanation: Given numbers : 3, 7, 12, α, $43 - \alpha$

Now, $\quad \bar{x} = \frac{3 + 7 + 12 + \alpha + 43 - \alpha}{}$

$$\Rightarrow \quad \bar{x} = 13$$

$$\because \quad \text{variance} = \frac{\Sigma x_1^2}{N} - (\bar{x})^2$$

$$\Rightarrow \text{Variance} = \frac{9 + 49 + 144 + \alpha^2 + (43 - \alpha)^2}{5} - (13)^2$$

$$\Rightarrow \text{Variance} = \frac{202 + \alpha^2 + \alpha^2 + 1849 - 86\alpha}{5} - 169$$

$$\Rightarrow \text{Variance} = \frac{2\alpha^2 - 86\alpha + 2051 - 845}{5}$$

$$\Rightarrow \text{Variance} = \frac{(2\alpha^2 - \alpha + 1) + (1205 - 85\alpha)}{5}$$

$$\Rightarrow \text{Variance} = \frac{(2\alpha^2 - \alpha + 1) + 5(241 - 17\alpha)}{5}$$

For variance to be natural number.

$$\frac{2\alpha^2 - \alpha + 1}{5} \in \mathrm{N}$$

$\Rightarrow$ $2\alpha^2 - \alpha + 1 - 5n = 0$ must have solution as natural number

Now, discriminant of above quadratic equation is

$$D = (-1)^2 - 4(2)(1 - 5n)$$

$$\Rightarrow \quad D = 40n - 7$$

So, D cannot be a perfect square as all perfect squares will be form of $4p$ or $4p + 1$ for $p \in N$

$\therefore$ α can't be natural number.

Hint:

$$\text{Variance} = \frac{\Sigma x_1^2}{N} - (\bar{x})^2$$

Shortcut:

$$\text{Mean } (\bar{x}) = \frac{3+7+12+\alpha+43-\alpha}{5} = 13$$

$$\text{Variance} = \frac{9+49+144+\alpha^2+(43-\alpha)^2}{5} - (13)^2 \in N$$

$$\Rightarrow \frac{2\alpha^2-\alpha+1}{5} \in N$$

$\Rightarrow \alpha^2 - \alpha + 1 - 5n = 0$ must have solution as natural number.

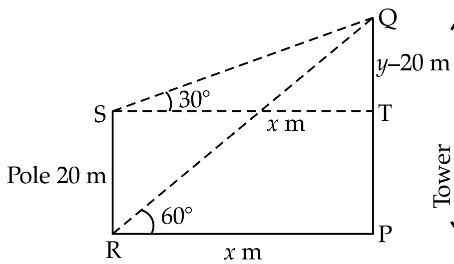
Now, its discriminant $D = 40n - 7$

$\Rightarrow$ D can't be perfect square as all perfect square wll be form of $4p$ or $4p + 1$ for $p \in N$.

$\Rightarrow \alpha$ can't be integer.

18. Option (D) is correct.

Explanation: Let height of the tower $PQ = y$ and distance between base of pole and base of tower $= x$



In ΔPQR, $\tan 60° = \frac{PQ}{PR} = \frac{y}{x}$

$\Rightarrow \sqrt{3} = \frac{y}{x}$

$\Rightarrow y = \sqrt{3}$ ...(i)

In ΔQTS, $\tan 30° = \frac{QT}{ST} = \frac{y-20}{x}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{y-20}{x}$

$\Rightarrow y = 20 + \frac{x}{\sqrt{3}}$

$\Rightarrow y = 20 + \frac{y}{3}$

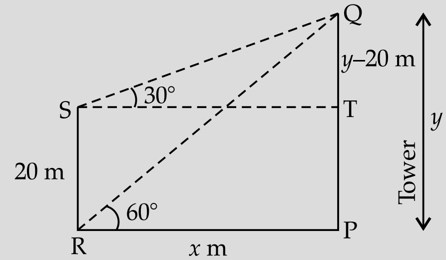
$\Rightarrow \frac{2y}{3} = 20$

$\Rightarrow y = 30$ m.

Hint:

Draw the diagram as per question and then apply the concept of height and distance.

Shortcut:



From, figure $\frac{y-20}{x} = \frac{1}{\sqrt{3}}$ and $\frac{y}{x} = \sqrt{3}$

$\Rightarrow y = 30$ m.

19. Option (C) is correct.

Explanation: The given statement is $(P \vee q)$

$\Rightarrow (\sim r \vee P)$

$\equiv \sim (P \vee q) \vee (\sim r \vee P)$

Now, let us take negation of above statement

$\equiv \sim [\sim(P \vee q) \vee (\sim r \vee P)]$

$\equiv \sim (\sim(P \vee q) \wedge \sim (\sim r \vee P)$

{using De Margen's law}

$\equiv (P \vee q) \wedge (\sim(\sim r) \wedge \sim P)$ $\{\sim (\sim a) \equiv a\}$

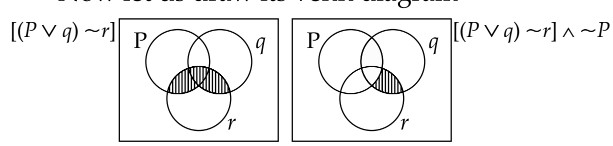
$\equiv (P \vee q) \wedge (r \wedge \sim P)$

$\equiv [(P \vee q) \wedge r] \wedge [(P \vee q) \wedge (\sim P)]$

{using distributive law}

$\equiv [(P \vee q) \wedge r] \wedge [\sim P]$

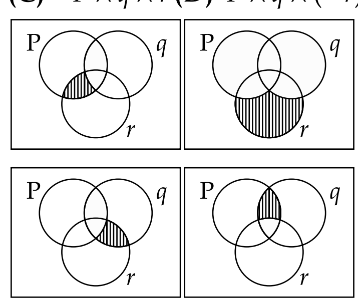
Now let us draw its venn diagram



Now let us draw Venn diagram of each option :

(A) $P \wedge (\sim q) \wedge r$ **(B)** $(\sim P) \wedge (\sim q) \wedge r$

(C) $\sim P \wedge q \wedge r$ **(D)** $P \wedge q \wedge (\sim r)$



So option (C) is correct.

Hint:

Simplify using De Morgan's law and distributive law and draw Venn Diagram

Shortcut:

$P \vee q \Rightarrow (\sim r \vee P) \equiv \sim (P \vee q) \vee (\sim r \vee P)$

$\equiv (\sim P \wedge \sim q) \vee (P \vee \sim r)$

$\equiv [(\sim P \vee P) \wedge (\sim q \vee P)] \vee \sim r$

$\equiv [\sim q \vee P] \vee \sim r$

Its negation is $\sim P \wedge q \wedge r$

20. Option (C) is correct.

Explanation: Given: $9^n - 8n - 1 = 64\,\alpha$ and $6^n - 5n - 1 = 25\,\beta$

$\Rightarrow$ so, $\alpha = \dfrac{9^n - 8n - 1}{64}$

$\Rightarrow \alpha = \dfrac{(1+8)^n - 8n - 1}{64}$

$\Rightarrow \alpha = \dfrac{{}^nc_0 + {}^nc_1 8 + {}^nc_2\, 8^2 + \ldots\ldots + {}^nc_n\, 8^n - 8n - 1}{64}$

$\Rightarrow \alpha = \dfrac{1 + 8n + {}^nc_2\, 8^2 + \ldots\ldots + {}^nc_n\, 8^n - 8n - 1}{64}$

$\Rightarrow \alpha = {}^nc_2 + {}^nc_3\, 8 + \ldots\ldots + {}^nc_n\, 8^{n-2}$

And $\beta = \dfrac{6^n - 5n - 1}{25}$

$\Rightarrow \beta = \dfrac{(1+5)^n - 5n - 1}{25}$

$\Rightarrow \beta = {}^nc_2 + {}^nc_3\, 5 + \ldots\ldots + {}^nc_n\, 5^{n-2}$

Now, $\alpha - \beta\ {}^nc_3\,(8-5) + {}^nc_4\,(8^2 - 5^2) + \ldots\ldots + {}^nc_n\,(8^{n-2} - 5^{n-2})$

Shortcut:

Given : $9^n - 8n - 1 = 64\,\alpha$ and $6n - 5n - 1\ 25\,\beta$

$\Rightarrow \quad \alpha = \dfrac{(1+8)^n - 8n - 1}{64}$

and $\quad \beta = \dfrac{(1+5)^n - 5n - 1}{25}$

$\Rightarrow \quad \alpha = {}^nc_2 + {}^nc_3\, 8 + \ldots\ldots + {}^nc_n\, 8^{n-2}$

and $\quad \beta = {}^nc_2 + {}^nc_3\, 5 + \ldots\ldots + {}^nc_n\, 5^{n-2}$

$\Rightarrow \quad \alpha - \beta = {}^nc_3\,(8-5) + {}^nc_4\,(8^2 - 5^2) + \ldots\ldots + {}^nc_n\,(8^{n-2} - 5^{n-2})$

Section B

21. Note: Students should be awarded the marks who has attempted this question.

Explanation: $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$

Now, $\vec{a}\cdot\vec{b} = 1 - 2 + 3 = 2 \quad \ldots(i)$

Also, given $\vec{a} + (\vec{b} \times \vec{c}) = 0$

$\Rightarrow \quad \vec{a} = -(\vec{b} \times \vec{c})$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = -(\vec{b} \times \vec{c})\cdot\vec{b} \quad \ldots(ii)$

Equation (i) and equation (ii) are contradicting.

Hint:

Find $\vec{a}\cdot\vec{b}$ using dot product formula and also find $\vec{a}\cdot\vec{b}$ using given education $\vec{a} + (\vec{b} \times \vec{c})$ and analyse both result.

Shortcut:

$\vec{a}\cdot\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})\cdot(\hat{i} + \hat{j} + \hat{k}) = 2 \quad \ldots(i)$

Given: $\vec{a} + (\vec{b} \times \vec{c}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = 0 \quad \ldots(ii)$

Equation (i) and equation (ii) are contradicting

22. Correct answer is [14]

Explanation: $(x-1)\dfrac{dy}{dx} + 2xy = \dfrac{1}{x-1}$

It is linear differential equation.

$\Rightarrow \quad \dfrac{dy}{dx} + \dfrac{2x}{x-1}y = \dfrac{1}{(x-1)^2}$

Comparing above equation with $\dfrac{dy}{dx} + py = Q$, we get

$P = \dfrac{2x}{x-1}y$ and $Q = \dfrac{1}{(x-1)^2}$

Now, $\quad I.F. = e^{\int P.dx}$

$\Rightarrow \quad I.F. = e^{\int \frac{2x}{x-1}dx}$

$\Rightarrow \quad I.F. = e^{\int\left(2 + \frac{2}{x-1}\right)dx}$

$\Rightarrow \quad I.F. = e^{2x + 2\ln|x-1|}$

$\Rightarrow \quad I.F. = e^{2x}.\ e^{\ln(x-1)^2}$

$\Rightarrow \quad I.F. = (x-1)^2 e^{2x}$

So, solution of given differential equation is given by

$$y\,(I.F.) = \int Q.\,(I.F.)\,dx$$

$$\Rightarrow \quad ye^{2x}(x-1)^2 = \int \frac{1}{(x-1)^2} e^{2x}(x-1)^2\,dx$$

$$\Rightarrow \quad ye^{2x}(x-1)^2 = \int e^{2x}\,dx$$

$$\Rightarrow \quad ye^{2x}(x-1)^2 = \frac{e^{2x}}{2} + C$$

Put $x = 2$ in above equation, we get

$$y(2)\,e^4 = \frac{e^4}{2} + C$$

$$\Rightarrow \quad C = \frac{1+e^4}{2e^4}.e^4 - \frac{e^4}{2} \quad \left\{\because\ y(2) = \frac{1+e^4}{2e^4}\right\}$$

$$\Rightarrow \quad C = \frac{1}{2}$$

$$\Rightarrow \quad ye^{2x}(x-1)^2 = \frac{e^{2x}}{2} + \frac{1}{2}$$

Put $x = 3$ in above equation, we get

$$y\,(3)\,e^6\,4 = \frac{e^6}{2} + \frac{1}{2}$$

$$\Rightarrow \quad \frac{e^{\alpha}+1}{\beta e^{\alpha}} = \frac{e^6+1}{8e^6}$$

$$\Rightarrow \quad \alpha = 6, \beta = 8$$

$$\therefore \quad \alpha + \beta = 14$$

Hint:

Solution of linear differential equation $\frac{dy}{dx} + py = Q$ is given by $y\,(I.F.) = \int Q.\,(I.F.) + C$, where $I.F. = e^{\int P.dx}$.

Shortcut:

Given: $(x-1)\frac{dy}{dx} + 2xy = \frac{1}{x-1}$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x-1}y = \frac{1}{(x-1)^2}$$

Now, $\quad I.F. = e^{\int \frac{2x}{x-1}dx} = e^{2x}(x-1)^2$

So, solution of given differential equation is given by

$$ye^{2e}(x-1)^2 = \int \frac{1}{(x-1)^2} e^{2x}(x-1)^2\,dx$$

$$= \frac{e^{2x}}{2} + C$$

$$\because \quad y(2) = \frac{1+e^4}{2e^4}$$

$$\Rightarrow \quad C = \frac{1}{2}$$

$$\because \quad y(3) = \frac{e^{\alpha}+1}{\beta e^{\alpha}}$$

$$\Rightarrow \quad \frac{e^6+1}{8e^6} = \frac{e^{\alpha}+1}{\beta e^{\alpha}}$$

$$\Rightarrow \quad \alpha = 6, \beta = 8 \Rightarrow \alpha + \beta = 14$$

23. Correct answer is [2223]

Explanation: Given: 3, 6, 9, 12, 15, 17, 21……up to 78 terms

5, 9, 13, 17, ……up to 59 terms

Now, last term of first series

$$t_{78} = 3 + 77 \times 3 = 234$$

And last term of second series

$$t_{59} = 5 + 58 \times 4 = 237$$

Now, common difference of common terms

$$= LCM\,\{3, 4\} = 12$$

$\therefore$ First common term is 9 and last common term is 225.

So, series will be 9, 21, 33, ……, 225

$$\because \quad t_n = 225$$

$$\Rightarrow \quad 9 + (n-1)\,12 = 225$$

$$\Rightarrow \quad n = 19$$

$\therefore$ sum of common on terms $= S_n = \frac{19}{2}(9 + 225) = 2223$

Hint:

(i) Find last term of both the given series and get idea about last term of required series.

(ii) n^{th} term of $A.P.$ is given by $t_n = a + (n-1)\,d$, where a = first term and d = common difference.

(iii) Sum of n terms of $A.P.$ $= \frac{n}{2}$ (First term + last term)

Shortcut:

S_1 : 3, 6, 9, 12, ……up to 78 terms

S_2 : 5, 9, 13, ……up to 59 terms

Common terms are 9, 21,…

T_{78} of $S_1 = 3 + 77 \times 3 = 234$

And last term of second series

T_{59} of $S_2 = 5 + 58 \times 4 = 237$

So, n^{th} common term ≤ 234

$\Rightarrow \quad n < \frac{237}{12}$

$\Rightarrow \quad n = 19$

So, $S_{19} = \frac{19}{2}[2(9) + 18 \times 12) = 2223$

24. Correct answer is [4]

Explanation:

Given, $\sin x = \cos^2 x$

$\Rightarrow \quad \sin x = 1 - \sin^2 x$

$\Rightarrow \quad \sin^2 x + \sin x - 1 = 0$

Let $\sin x = t$

$\Rightarrow \quad t^2 + t - 1 = 0$

$\Rightarrow \quad t = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$

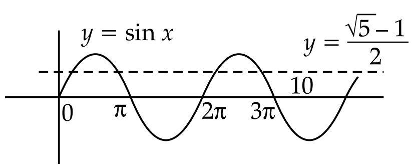
$\Rightarrow \quad t = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad \sin x = \frac{-1 \pm \sqrt{5}}{2}$

As we know $\sin x \in [-1, 1]$

$\therefore \quad \sin x = \frac{\sqrt{5} - 1}{2}$

Lets draw the graph of $y = \sin x$ and $y = \frac{\sqrt{5}-1}{2}$ for $x \in (0, 10)$



∵ Both curve: intersect at 4 points for $x \in (0, 10)$

∴ Number of solution for given equation in $x \in (0, 10)$ are 4.

Hint:

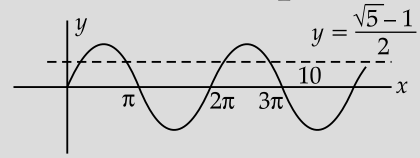
Simplify the given equation using $\cos^2 x = 1 - \sin^2 x$ and solve further.

Shortcut:

Given, $\sin x = \cos^2 x$

$\Rightarrow \quad \sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad \sin x = \frac{\sqrt{5} - 1}{2}$



∴ Number of solutions = 4

25. Correct answer is [12]

Explanation: Given:

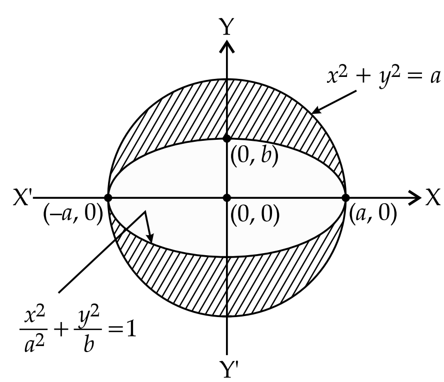
Area $\{(x, y) : x^2 + y^2 \le a^2 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} \ge 1\} = 30\pi$

And area $\{(x, y) : x^2 + y^2 \ge b^2 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$ $= 18\pi$

Case-1

$x^2 + y^2 \le a^2$ represents the region inside the circle $x^2 + y^2 = a^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} \ge 1$ represents the region outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



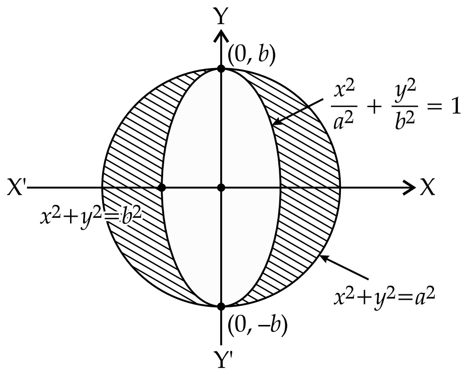
Now, shaded area $= \pi a^2 - \pi ab = 30\pi$

$\Rightarrow \quad a^2 = 30 + ab \quad \ldots(i)$

Case-2

$x^2 + y^2 \ge b^2$ represents the reigon outside the circle $x^2 + y^2 = b^2$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ represents the region inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Now, shared area $= \pi ab - \pi b^2 = 18\pi$

$\Rightarrow \quad b^2 = ab - 18 \quad \ldots$(ii)

Adding equation (i) and equation (ii), we get

$a^2 + b^2 = 2ab + 12$

$\Rightarrow \quad (a-b)^2 = 12$

Hint:

(i) Find shaded region for both area and solve further

(ii) Use area of circle $x^2 + y^2 = r^2$ is πr^2.

(iii) Use area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

Shortcut:

$\because$ Area $\{(x, y) : x^2 + y^2 \le a^2$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}\ge 1\}$ $= 30\pi$

$\Rightarrow \quad \pi a^2 - \pi ab = 30$

$\Rightarrow \quad a^2 = 30 + ab \quad \ldots$(i)

$\because$ Area $\{(x, y) : x^2 + y^2 \ge b^2$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$ $= 18\pi$

$\Rightarrow \quad \pi ab - \pi b^2 = 18$

$\Rightarrow \quad b^2 = ab - 18 \quad \ldots$(ii)

Adding equation (i) and equation (ii), we get

$a^2 + b^2 - 2ab = 12$

$\Rightarrow \quad (a-b)^2 = 12$

26. Correct answer is [4]

Explanation:

Given, $f\left(\frac{1}{4}\right) = f\left(\frac{1}{2}\right) = 0, f(1) = 1, g\left(\frac{3}{4}\right) = 0$ and $g(1) = 2$

Let $\quad p(x) = f(x)\, g'(x)$

$\Rightarrow \quad p'(x) = f'(x)\, g'(x) + f(x)\, g''(x)$

$\because f(x)$ is on even function

$\therefore \quad f\left(\frac{1}{4}\right) = f\left(-\frac{1}{4}\right) = f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = 0$

So, $f(x) = 0$ has minimum 4 roots

Also, given $g(x)$ is an even function

$\Rightarrow \quad g\left(\frac{3}{4}\right) = g\left(-\frac{3}{4}\right) = 0$

$\therefore \quad g(x) = 0$ has minimum 2 roots.

$\Rightarrow \quad g'(x) =$ has minimum one root.

So, $\quad p(x) = 0$ has minimum 5 roots.

$\Rightarrow \quad p'(x) = 0$ has minimum 4 roots.

Hint:

(i) $f(x) = 0$ has n roots, then $f'(x) = 0$ has minimum $n - 1$ roots.

(ii) Consider $p(x) = f(x)\, g'(x)$ and find $p'(x)$ and solve further.

Shortcut:

$\because f(x)$ is even

$\Rightarrow f\left(\frac{1}{4}\right) = f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = f\left(-\frac{1}{4}\right) = 0$

$\Rightarrow \quad f(x) = 0$ has minimum 4 roots

$\because g(x)$ is even

$\Rightarrow \quad g\left(\frac{3}{4}\right) = g\left(-\frac{3}{4}\right) = 0$

$\Rightarrow \quad g'(x) = 0$ has minimum one root.

Let $\quad p(x) = f(x)\, g'(x)$

$\Rightarrow \quad p'(x) = f'(x)\, g'(x) + f(x)\, g''(x)$

So, $\quad p(x) = 0$ has minimum 5 roots

$\Rightarrow \quad p'(x) = 0$ has minimum 4 roots

27. Correct answer is [5]

Explanation:

Given expansion is $\left(2x^{\frac{1}{5}} - \frac{1}{x^{1/2}}\right)^{15}$

Now, general term of given expansion is

$$T_{p+1} = (-1)^p\, {}^{15}C_p\, 2^{15-p} \left(x^{\frac{1}{5}}\right)^{15-p} . \left(\frac{1}{x^{1/5}}\right)^p$$

$$= (-1)^p\, {}^{15}C_p\, 2^{15-p} .\ x^{\frac{15-2p}{5}}$$

For coefficient of x^{-1}, $\frac{15-2p}{5} = -1$

$\Rightarrow \quad p = 10$

$\therefore \quad m = {}^{15}C_{10}\, 2^5$

For coefficent of x^{-3}, $\frac{15-2p}{5} = -3$

$\Rightarrow \quad p = 15$

$\therefore \quad n = -{}^{15}C_{15}\, 2^0 = -1$

Now, $\quad mn^2 = {}^{15}C_{10}\, 2^5$

$\Rightarrow \quad mn^2 = {}^{15}C_5\, 2^5$

$\Rightarrow \quad {}^{15}C_r\, 2^r = {}^{15}C_5\, 2^5$

$\Rightarrow \quad r = 5$

Hint:

General term of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Shortcut:

$$T_{p+1} = (-1)^p\, {}^{15}C_p\, 2^{15-p}\ x^{\frac{15-2p}{5}}$$

For x^{-1}, $p = 10$

$\therefore \quad m = {}^{15}C_{10}\, 2^5 = {}^{15}C_5\, 2^5$

For x^{-3}, $p = 15$

$\therefore \quad n = -1$

So, $\quad mn^2 = {}^{15}C_5\, 2^5 \Rightarrow r = 5$

28. Correct answer is [1086]

Explanation: Let *pqrs* is four digit number.

So, first three digit *pqr* should be divisible by '*s*'

If $s = 1$, then number of required 4 digit number = $9 \times 10 \times 10$

If $s = 2$, then number of required 4 digit number = $4 \times 5 \times 5$

If $s = 3$, then number of required 4 digit number = $3 \times 4 \times 4$

If $s = 4$, then number of required 4 digit number = $2 \times 3 \times 3$

If $s = 5$, then number of required 4 digit number = $1 \times 2 \times 2$

If $s = 6, 7, 8, 9$, then number of required 4 digit number = 4×4

$\therefore$ Total 4 digit required number = 900 + 100 + 48 + 18 + 4 + 16 = 1086

Hint:

Assume 4 digit number is *pqrs* and make cases for $s = 1$ to 9 and solve further using multiplication principle of counting.

29. Correct answer is [1]

Explanation:

Given, $\quad M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$

and $\quad N = \sum_{K=1}^{49} M^{2K}$

Now, $\quad M^2 = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$

$$= \begin{bmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{bmatrix}$$

$$= -\alpha^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -\alpha^2 I$$

Now, $\quad N = \sum_{K=1}^{49} M^{2K}$

$\Rightarrow \quad N = \sum_{K=1}^{49} (-\alpha^2 I)^K$

$\Rightarrow \quad N = \sum_{K=1}^{49} (-\alpha^2)^K\, I$

$\Rightarrow \quad N = (-\alpha^2)\left\{\frac{(-\alpha^2)^{49} - 1)}{-\alpha^2 - 1}\right\} I$

$$\Rightarrow \quad N = \frac{-\alpha^2(1+\alpha^{98})I}{1+\alpha^2}$$

Now, $I - M^2 = I + \alpha^2 I = (1+\alpha^2) I$

$\because \quad (I - M)^2 N = -2I$

$$\Rightarrow \quad (1+\alpha^2)\left(\frac{-\alpha^2(1+\alpha^{98})}{1+\alpha^2}\right) I = -2I$$

$\Rightarrow \quad \alpha^2(1+\alpha^{98}) = 2$

$\Rightarrow \quad \alpha = 1$

Hint:

Find M^2 using the concept of multiplication of matrices and find N using the formula of sum of n term of $G.P.$ and solve further.

Hint:

$$M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$$

$$\Rightarrow \quad M^2 = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{bmatrix} = -\alpha^2 I$$

Now, $N = [-\alpha^2 + \alpha^4 - \alpha^6 + \ldots]I$

$$= \frac{-\alpha^2(1+\alpha^{98})}{1+\alpha^2} I$$

$\therefore \quad (I - M^2)N = -\alpha^2(1+\alpha^{98}) = -2$

$\Rightarrow \quad \alpha = 1$

30. Correct answer is [18]

Explanation:

$f(g(x)) = 8x^2 - 2x$ and $g(f(x)) = 4x^2 + 6x + 1$

Let $f(x) = px^2 + qx + r$

And $g(x) = ux + v$

Now, $f(g(x)) = p(ux+v)^2 + q(ux+v) + r$

$= pu^2x^2 + (2puv + qu)x + pv^2 + qv + r$

$\Rightarrow pu^2 = 8,\ 2puv + qu = -2$ and $pv^2 + qv + r = 0$

Now, $g(f(x)) = u(px^2 + qx + r) + v$

$= pux^2 + qux + v + ur$

$\Rightarrow \quad pu = 4, qu = 6, v + ur = 1$

$\therefore \quad u = 2, q = 3, p = 2, v = -1, r = 1$

$\therefore \quad f(x) = 2x^2 + 3x + 1$

& $\quad g(x) = 2x - 1$

Now, $f(2) = 2(2)^2 + 3(2) + 1 = 15$

& $\quad g(2) = 2(2) - 1 = 3$

$\therefore \quad f(2) + g(2) = 18$

Hint:

Let $f(x) = px^2 + qx + r$ and $g(x) = ux + v$ and find $f(g(x))$, $g(f(x))$ using the concept of composition function and solve further.

Shortcut:

Given: $f(g(x)) = 8x^2 - 2x$

$g(f(x)) = 4x^2 + 6x + 1$

So, $f(x) = 2x^2 + 3x + 1$ and $g(x) = 2x - 1$

Now, $f(2) = 8 + 6 + 1 = 15$ and $g(2) = 3$

$\therefore f(2) + g(2) = 18$

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